EPFL

Prof. Anastasios P. Vassilopoulos

Advanced composites in engineering structures
Lecture III – Mechanics of composites

GRoMeC
Composite Mechanics Group

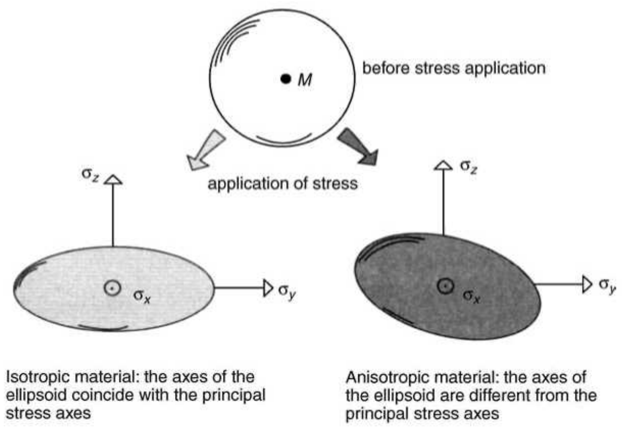
Homogeneous, Isotropic and anisotropic material

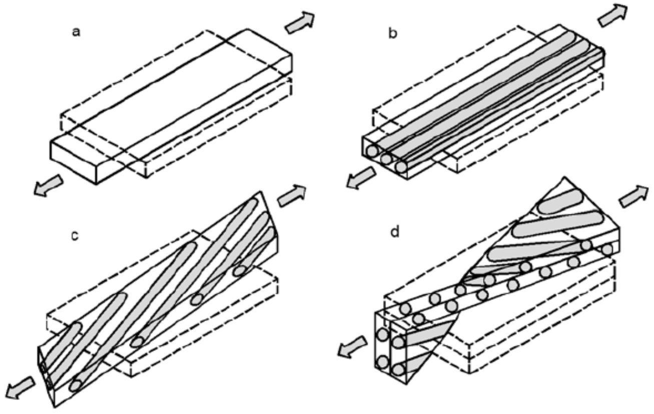
Homogeneous: Same properties at any site

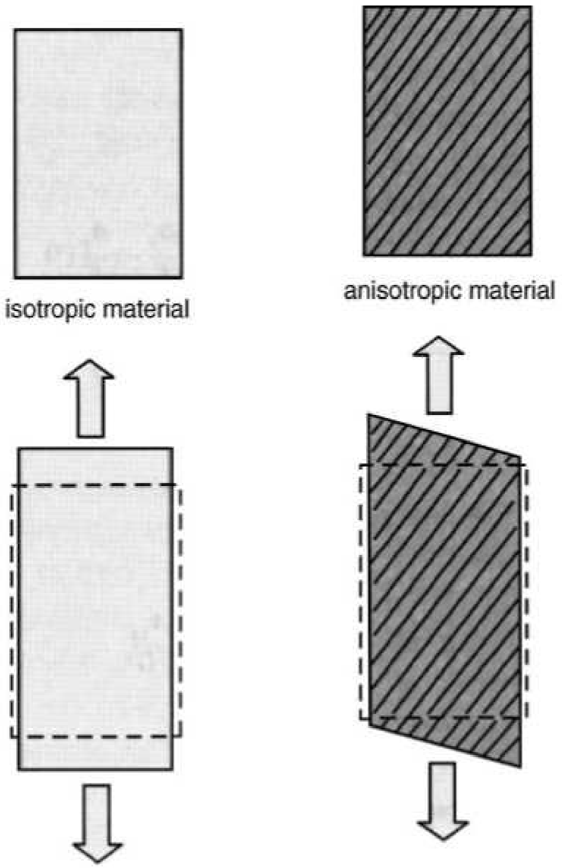
Isotropic: Same properties at all directions

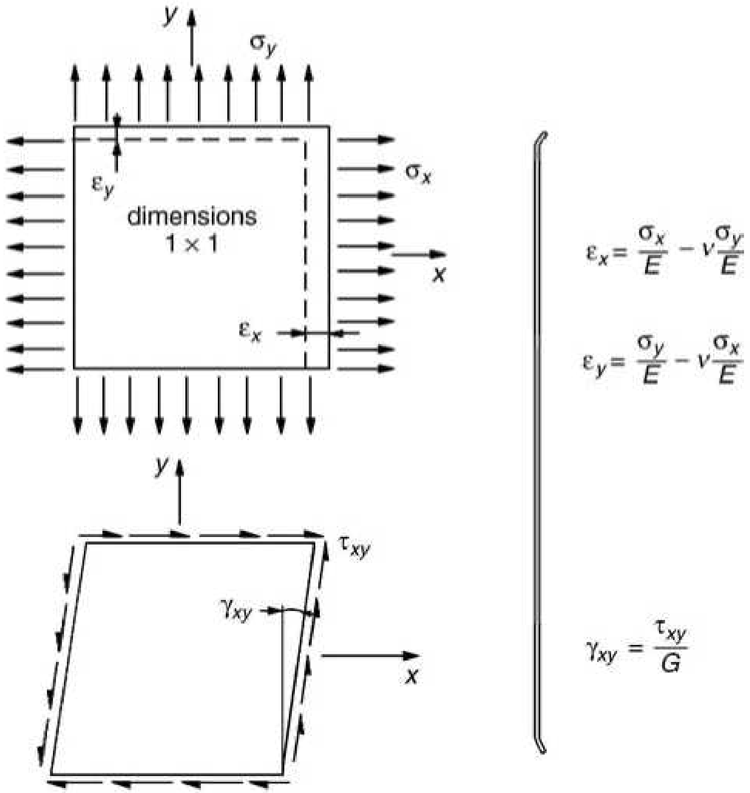
Anisotropic: Different properties along different directions

Low or high anisotropy…


before stress application
M
application of stress
σz
σy
σx
Isotropic material: the axes of the ellipsoid coincide with the principal stress axes
Anisotropic material: the axes of the ellipsoid are different from the principal stress axes


a
b
c
d


isotropic material
anisotropic material



ISOTROPIC

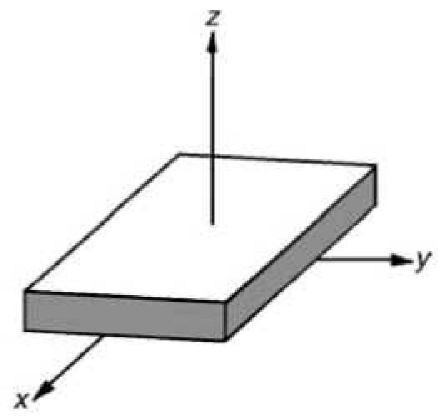
MATERIAL

3 Elastic constants exist…E, ν, G

However, since:

$$G = \frac{E}{2(1+\nu)}$$

Only 2 INDEPENDENT elastic constants are necessary to characterize an elastic isotropic material.



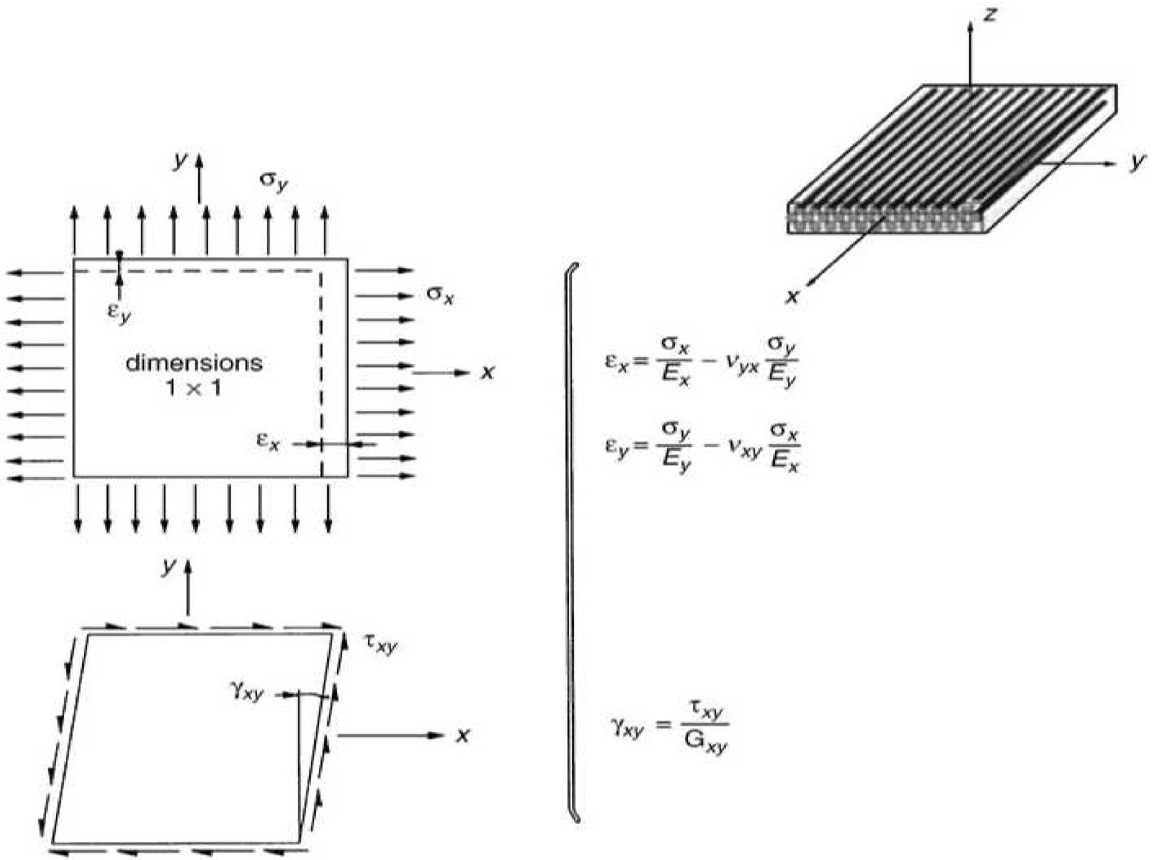
Stress-strain relation for elastic-isotropic materials

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{v}{E} & 0 \\ -\frac{v}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are the small strains

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \varepsilon_y = \frac{\partial u_y}{\partial y}, \text{ and } \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

EPFL



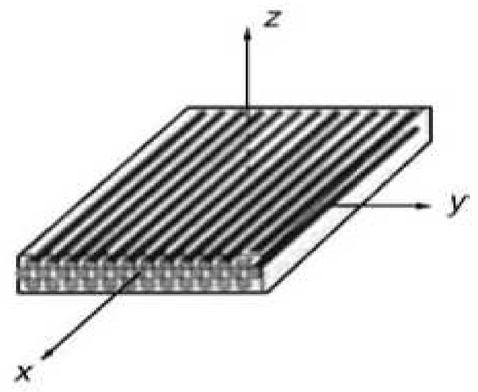
ANISOTROPIC MATERIAL

(here) Unidirectional lamina

- 5 distinct elastic constants:
  - Two moduli of elasticity: E_x, E_y
  - Two Poisson coefficients: ν_{xy}, ν_{yx}
  - One shear modulus: G_{xy}

symmetry →

$$\nu_{xy} = \nu_{yx}\frac{E_x}{E_y}$$



Stress-strain relation for elastic-anisotropic material (plane stress)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{v_{yx}}{E_y} & 0 \\ -\frac{v_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are the small strains

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \varepsilon_y = \frac{\partial u_y}{\partial y}, \text{ and } \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Composites…

- Homogeneous Isotropic (conventional) materials…
- Composite materials
  - Homogeneous **an**isotropic material
  - Linear elastic up to failure (superposition, load-unload, no time/rate dependency)
  - Hooke's Law

→ Axiom (principle)?

→ Based on physical background?

→ Empirical equation?

Composites…

- Homogeneous Isotropic (conventional) materials…
- Composite materials
  - Homogeneous **<u>an</u>**isotropic material
  - Linear elastic up to failure (superposition, load-unload, no time/rate dependency)
  - Hooke's Law

→ Axiom (principle)?

→ Based on physical background?

→ Empirical equation?

Hooke's Law

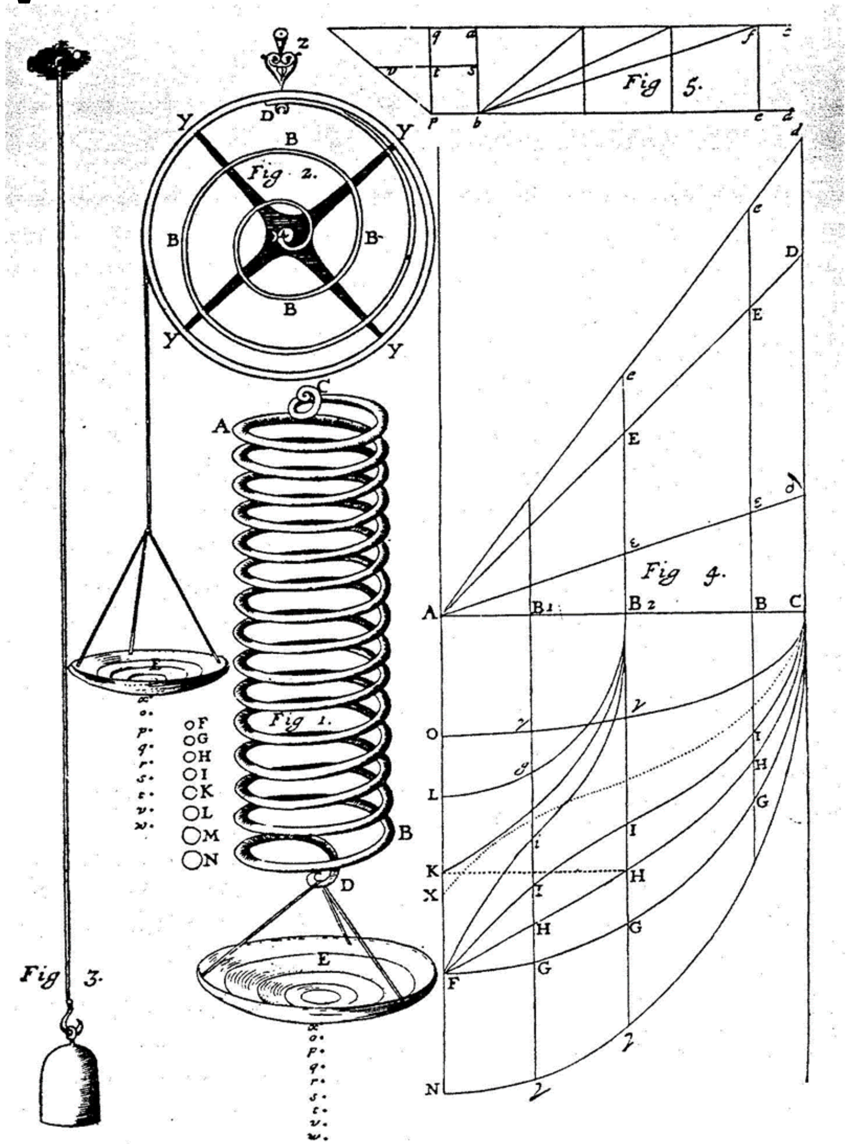
Robert Hooke (1635-1703)

"De Potentia restitutivâ" or "Of Spring" (1678)

Anagram: C E I I I N O S S S T T U V

"UT TENSIO SIC VIS"

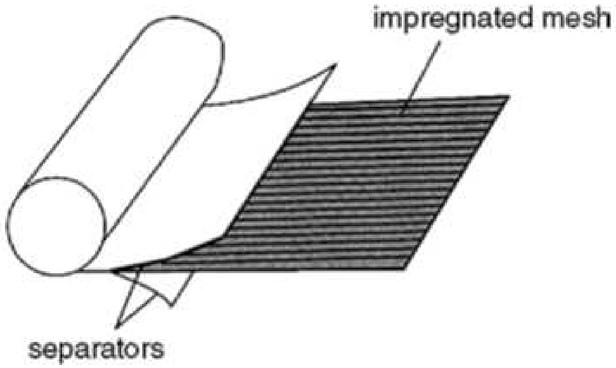
"As the extension, so the force"

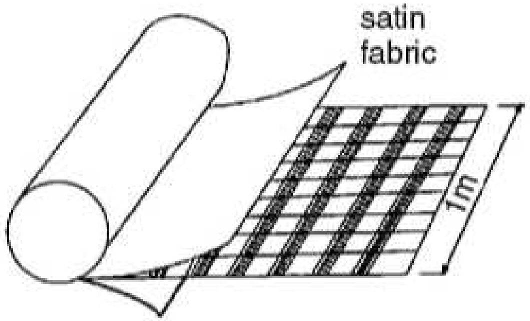


EPFL

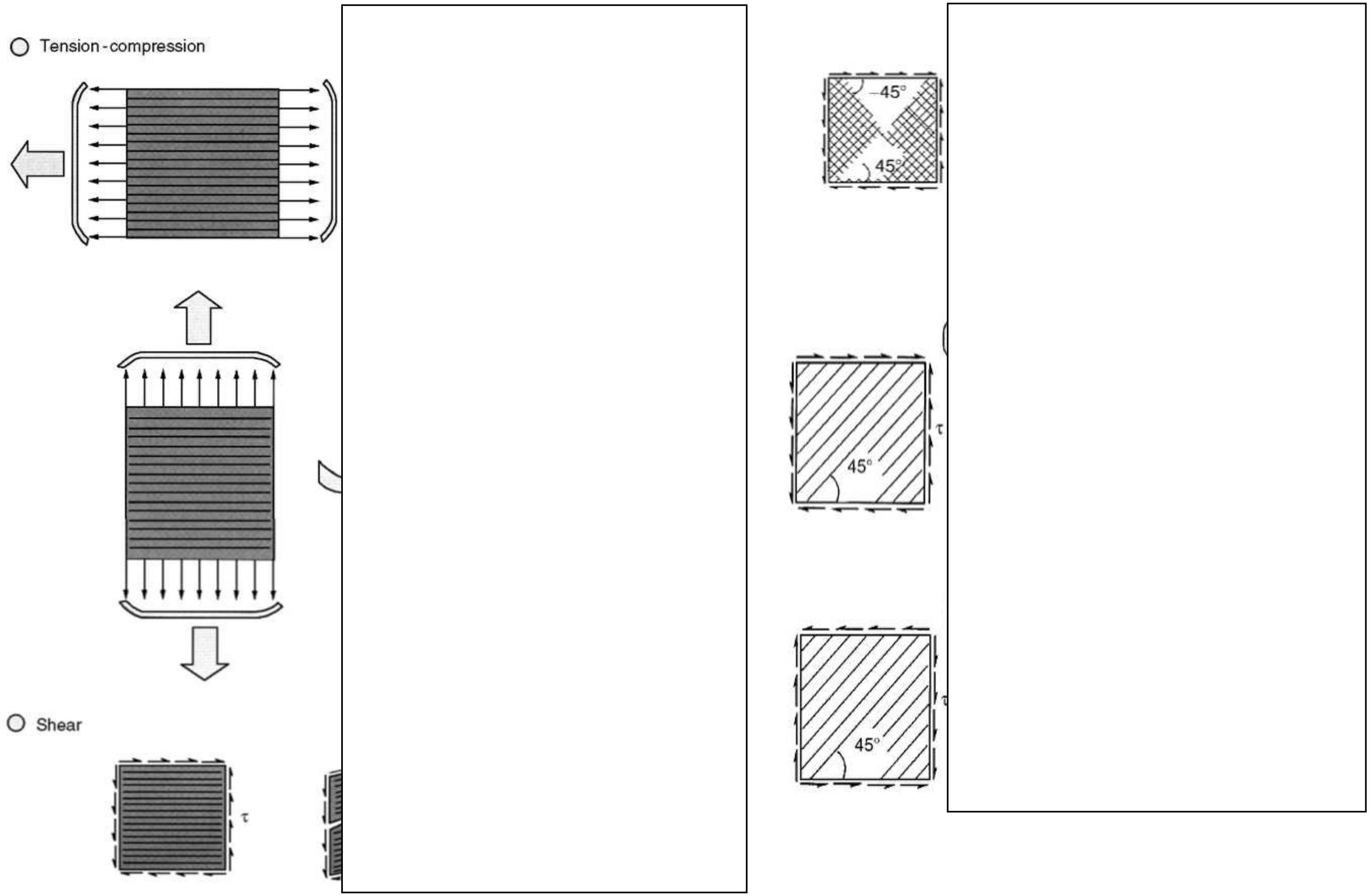
The laminate

A laminate results in the superposition of many layers, or plies, or sheets, made of unidirectional layers, fabrics or mats with proper orientation in each ply





Importance of ply orientation



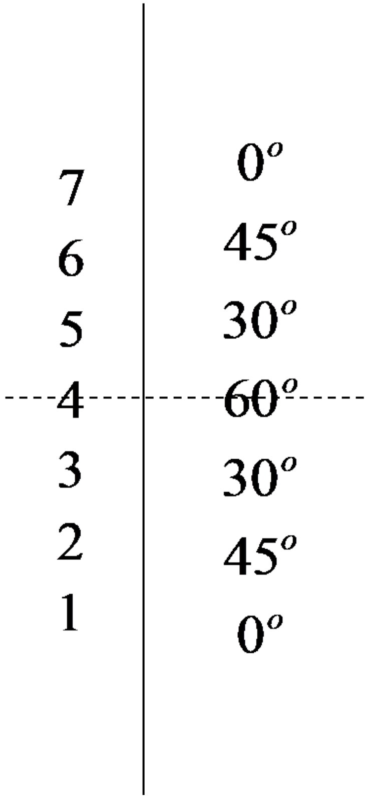
Code to represent a laminate

- Each ply is noted by its orientation
- The successive plies are separated by a slash
- An index is used to indicate more than one successive plies along the same orientation
- Indices T or S are placed at the end to determine Total, or Symmetric laminates
- Bar, e.g., $\overline{90}$ is used in symmetric laminates when plane of symmetry is <u>in</u> the current layer, e.g., laminate $[0/90/0]_T$ could also be written as: $[0/\overline{90}]_S$

Code to represent laminates
Examples

10	90^o	
9	0^o	
8	0^o	
7	45^o	
6	-45^o	$[90/0_2/45/-45/-45/45/0_2/90]_T$
5	-45^o	$[90/0_2/45/-45]_S$
4	45^o	
3	0^o	
2	0^o	
1	90^o	

Code to represent laminates
Examples



$$[0/45/30/60/30/45/0]_T$$

$$[0/45/30/\overline{60}]_S$$

Code to represent laminates

90^o
0^o
45^o
-45^o
90^o

90^o
45^o
-45^o
-45^o
45^o
90^o

0^o
90^o
$\pm 45^o$
90^o
0^o

$[0/90/\pm 45/90/0]_T$

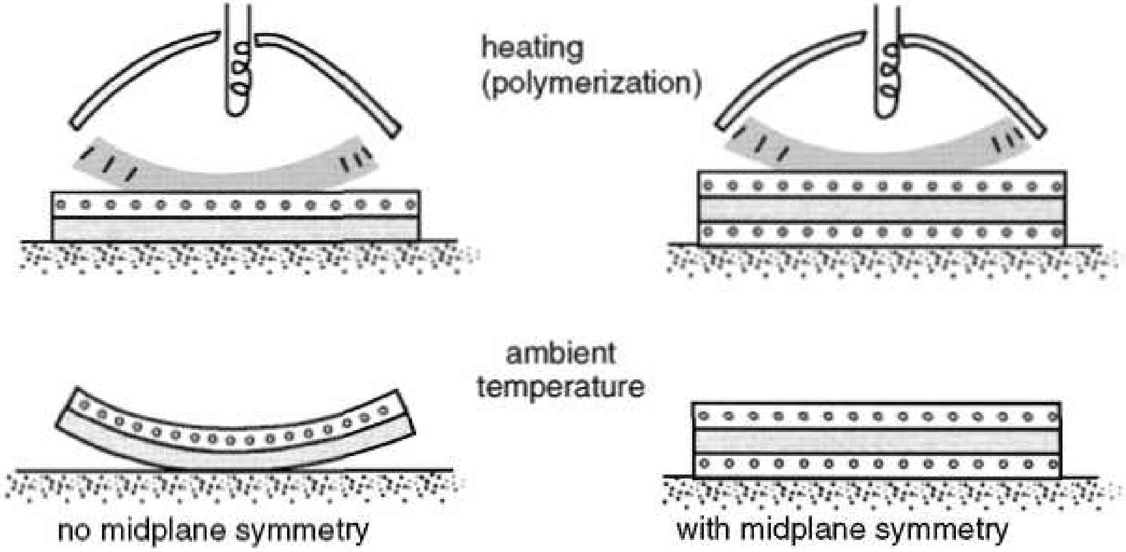
$[90 / -45 / 45 / 0 / 90]_T$

$[90 / 0 / 45 / -45 / 90]_T$

$[0 / 90 / \overline{\pm 45}]_S$

$[90 / 45 / -45]_S$

$[90 / 45 / -45 / -45 / 45 / 90]_T$



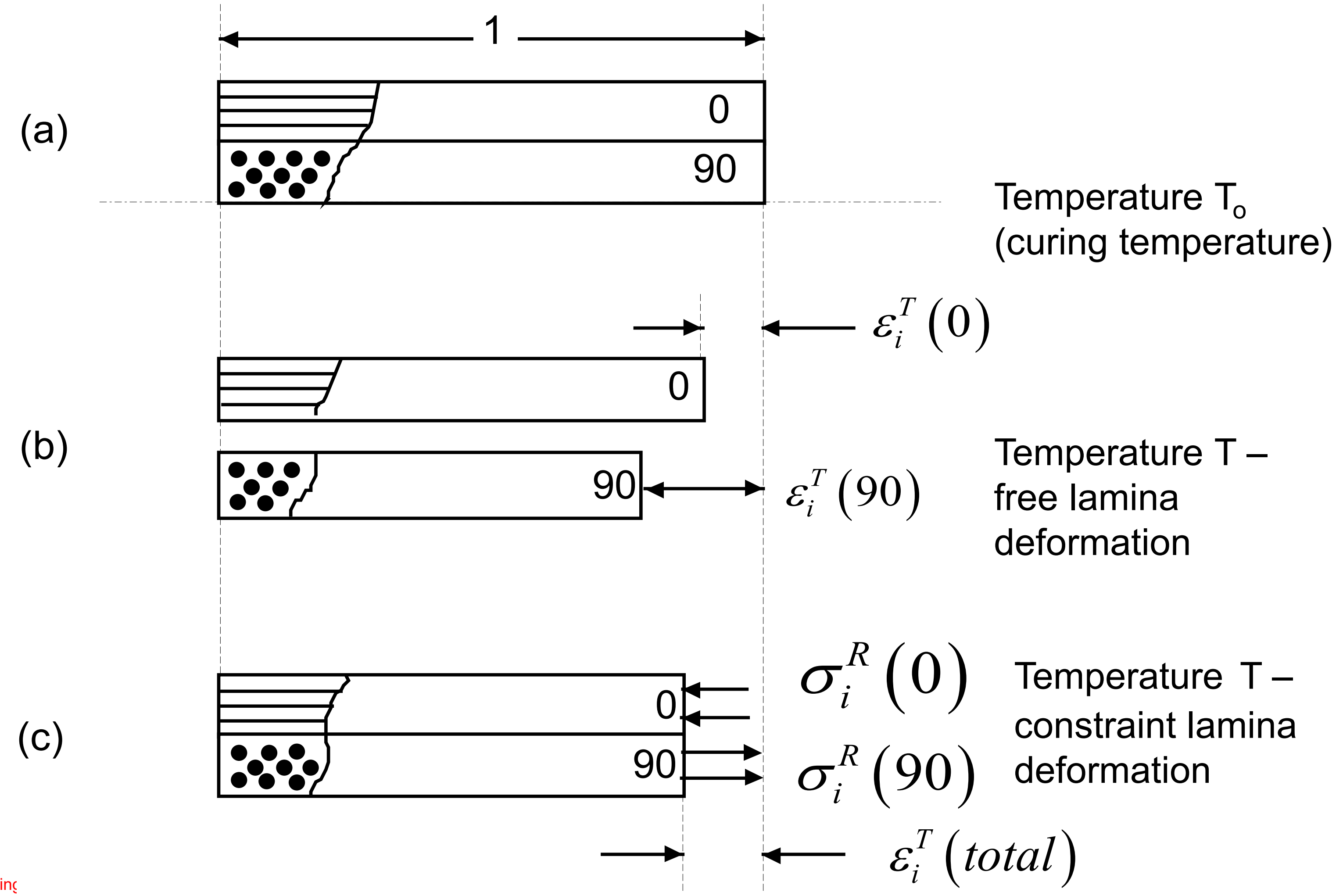
Need for mid-plane symmetry

Mid-plane symmetry prevents the deformations due to thermal residual stresses…

Mid-plane symmetry imposes the symmetry of these stresses and prevents the deformations of the whole part.

Remaining stresses

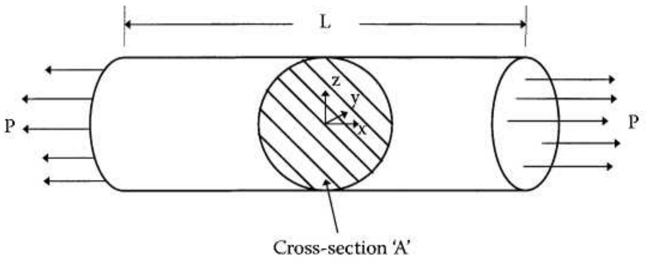
Residual stresses during fabrication, e.g. $[0/90]_S$



Macromechanical analysis of a lamina

- For a linear isotropic material in a three dimensional stress state, the Hooke's law stress strain relationships at a point in a x-y-z orthogonal system can be written as:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix},$$



Determine stress and strain state

$$\sigma_x = \frac{P}{A}, \sigma_y = 0, \sigma_z = 0, \tau_{yz} = 0, \tau_{zx} = 0, \tau_{xy} = 0.$$

$$\varepsilon_x = \frac{P}{AE}, \varepsilon_y = -\frac{\nu P}{AE}, \varepsilon_z = -\frac{\nu P}{AE},$$

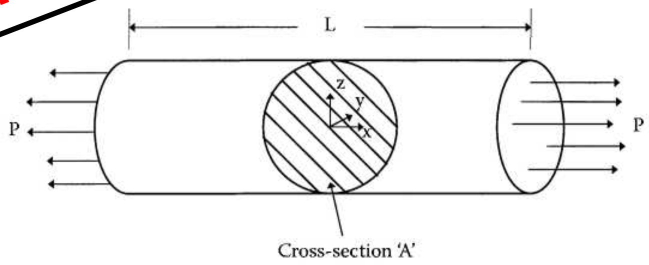
$$\gamma_{yz} = 0, \gamma_{zx} = 0, \gamma_{xy} = 0.$$

Macromechanical analysis of a lamina

- For a linear isotropic material in a th l stress state, the Hooke's la hips at a point in a x-y-z orthog written as:

Interaction of normal stresses with shear strains and of shear stresses with normal strains is missing!

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} =
\begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix},
$$



Determine stress and strain state

$$\sigma_x = \frac{P}{A}, \sigma_y = 0, \sigma_z = 0, \tau_{yz} = 0, \tau_{zx} = 0, \tau_{xy} = 0.$$

$$\varepsilon_x = \frac{P}{AE}, \varepsilon_y = -\frac{\nu P}{AE}, \varepsilon_z = -\frac{\nu P}{AE},$$

$$\gamma_{yz} = 0, \gamma_{zx} = 0, \gamma_{xy} = 0.$$

Hooke's Law for different types of materials

- For a general material, linearly elastic and anisotropic, the general stress-strain relationship is expressed as:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix},$$

Matrix [C] is called the stiffness matrix.

In this general case it contains 36 constants.

$C_{ij}=C_{ji}$ and thus only 21 independent variables exist

1-2-3, 1´-2´-3´ systems…

Hooke's Law

$$\sigma_{I,II} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\varepsilon_{I,II} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Notation:

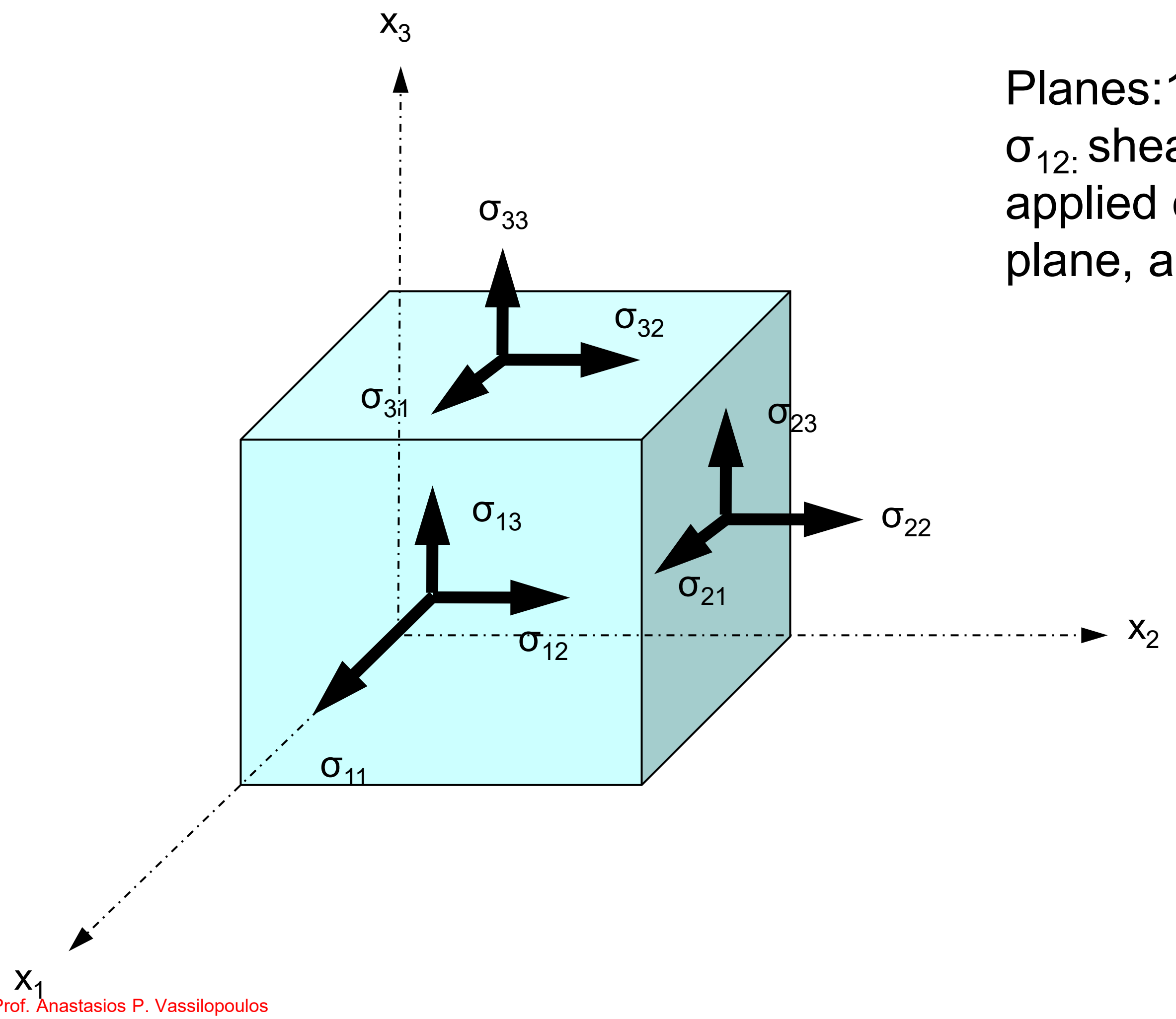
$$\begin{array}{ccccccc} 11 & \rightarrow & 1 & \quad & 23 & \rightarrow & 4 \\ 22 & \rightarrow & 2 & & 13 & \rightarrow & 5 \\ 33 & \rightarrow & 3 & & 12 & \rightarrow & 6 \end{array}$$

Hooke's law in matrix format:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \qquad \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \qquad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

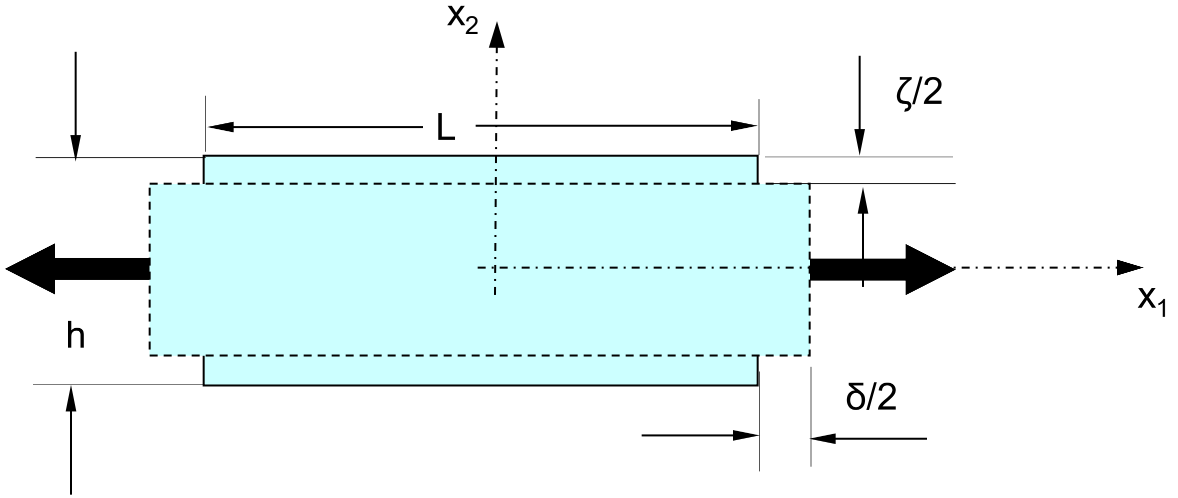
Not tensorial components but technical shear strains: e.g., $\varepsilon_4 = \gamma_{23}$

Geometrical meaning of the stress tensor components

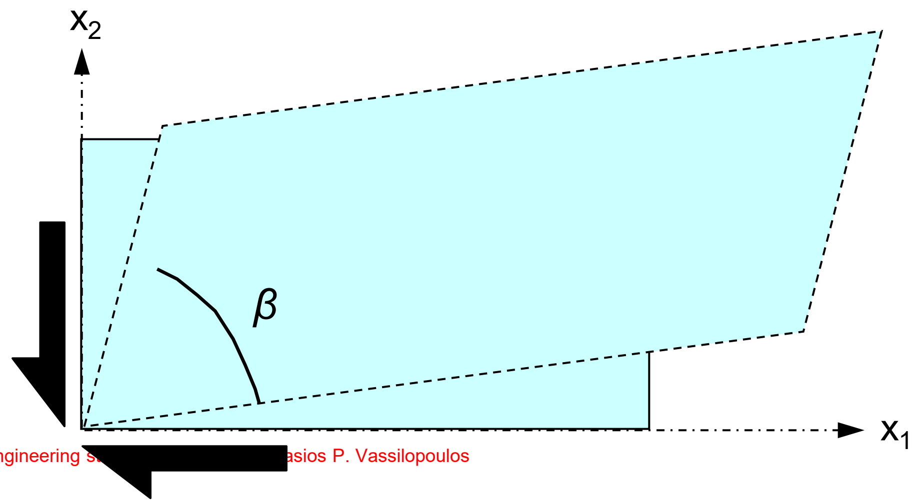


Planes:12, 13, 23
$\sigma_{12:}$ shear stress applied on the 23 plane, along axis 2.

Geometrical meaning of the strain components



$$\varepsilon_{11} = \frac{\delta}{L}, \quad \varepsilon_{22} = \frac{\zeta}{h}$$



$$\gamma_{12} = 2\varepsilon_{12} = \frac{\pi}{2} - \beta, \quad (\beta \text{ in rad})$$

Compliance matrix

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} .$$

And for the case of an isotropic material:

$$S_{11} = \frac{1}{E} = S_{22} = S_{33}$$

$$S_{12} = -\frac{\nu}{E} = S_{13} = S_{21} = S_{23} = S_{31} = S_{32} \, ,$$

$$S_{44} = \frac{1}{G} = S_{55} = S_{66} \, ,$$

All other S_{ij} components equal to zero.

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} ,$$

Compliance and stiffness matrix are always symmetric

The stress strain relationship can be expressed as:

$$\sigma_i = \sum_{j=1}^{6} C_{ij}\varepsilon_j, \qquad i = 1\ldots6$$

In this contracted notation:

$$\sigma_4 = \tau_{23},\ \sigma_5 = \tau_{31},\ \sigma_6 = \tau_{12},$$
$$\varepsilon_4 = \gamma_{23},\ \varepsilon_5 = \gamma_{31},\ \varepsilon_6 = \gamma_{12}$$

The strain energy in the body per unit volume is:

$$W = \frac{1}{2}\sum_{i=1}^{6}\sigma_i\varepsilon_i,$$

By substituting Hooke's law in the previous equation:

$$W = \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} C_{ij}\varepsilon_i\varepsilon_j,$$

And by partial differentiation of W:

$$\frac{\partial W}{\partial\varepsilon_i\,\partial\varepsilon_j} = C_{ij}, \qquad \frac{\partial W}{\partial\varepsilon_j\,\partial\varepsilon_i} = C_{ji}$$

Which means that:

$$C_{ij} = C_{ji}$$

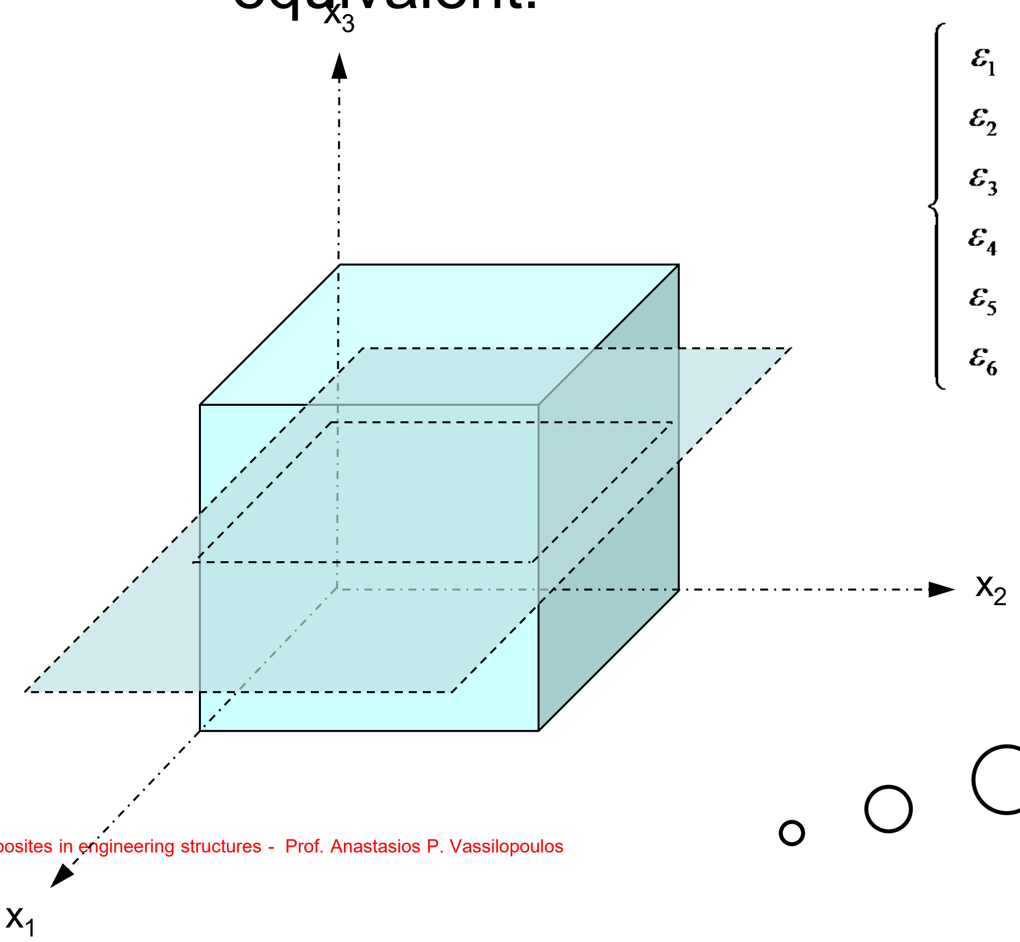
Because differentiation does not necessarily need to be in order!

Anisotropic material

- The material that has 21 independent elastic constants at a point
  - After determination of these constants for a specific **point**, then stress-strain relations could be developed, for that **point**.
  - If these constants are the same at different **points** of the material, then the material is called homogeneous, in other case, it is called non-homogeneous.
  - These 21 constants should be determined experimentally
  - However, some materials have same properties in different directions because of symmetry in their internal structure

Monoclinic material

- Material with one plane of symmetry. Directions symmetric with respect to that plane are elastically equivalent.



$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16}\\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26}\\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36}\\ 0 & 0 & 0 & S_{44} & S_{45} & 0\\ 0 & 0 & 0 & S_{45} & S_{55} & 0\\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}$$

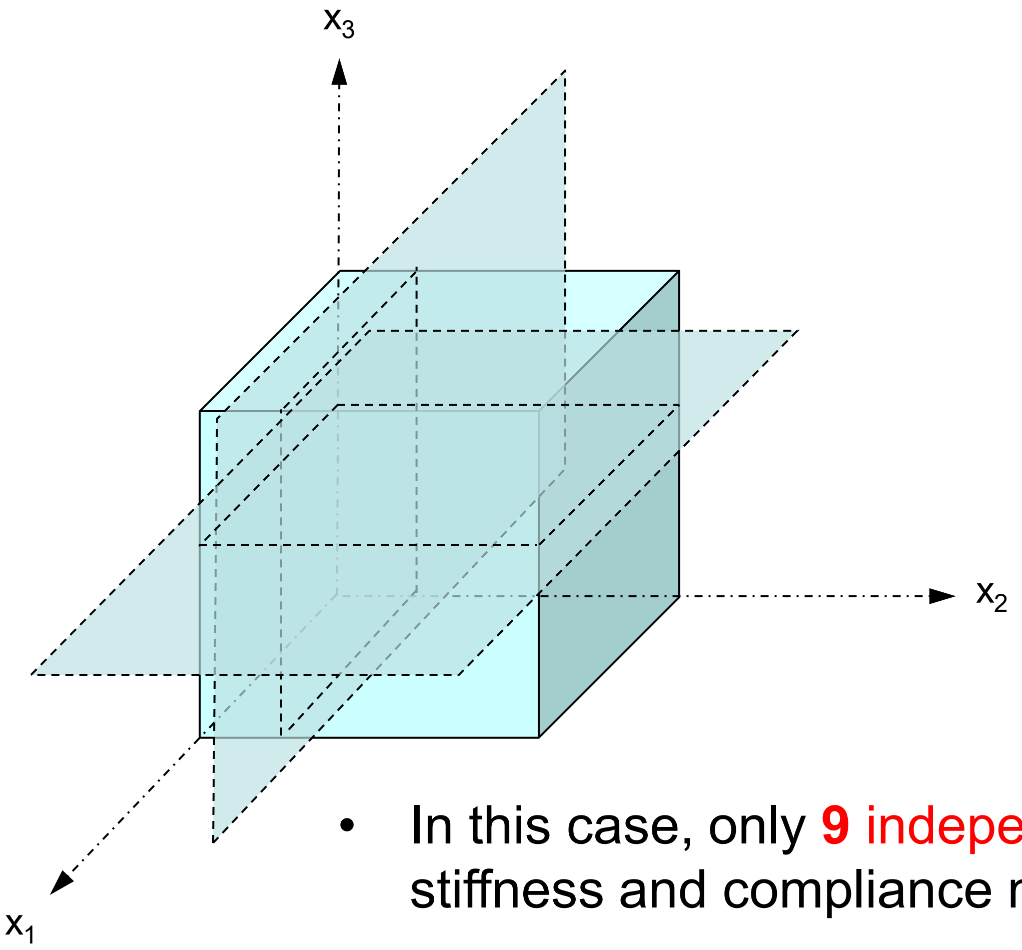
Stiffness matrix has this form only because the plane x_1-x_2 has been selected as the symmetry plane…

$$C_{14}=0, C_{15}=0, C_{24}=0, C_{25}=0, C_{34}=0, C_{35}=0, C_{46}=0, C_{56}=0.$$

EPFL

Orthotropic material

- Material with two perpendicular planes of symmetry

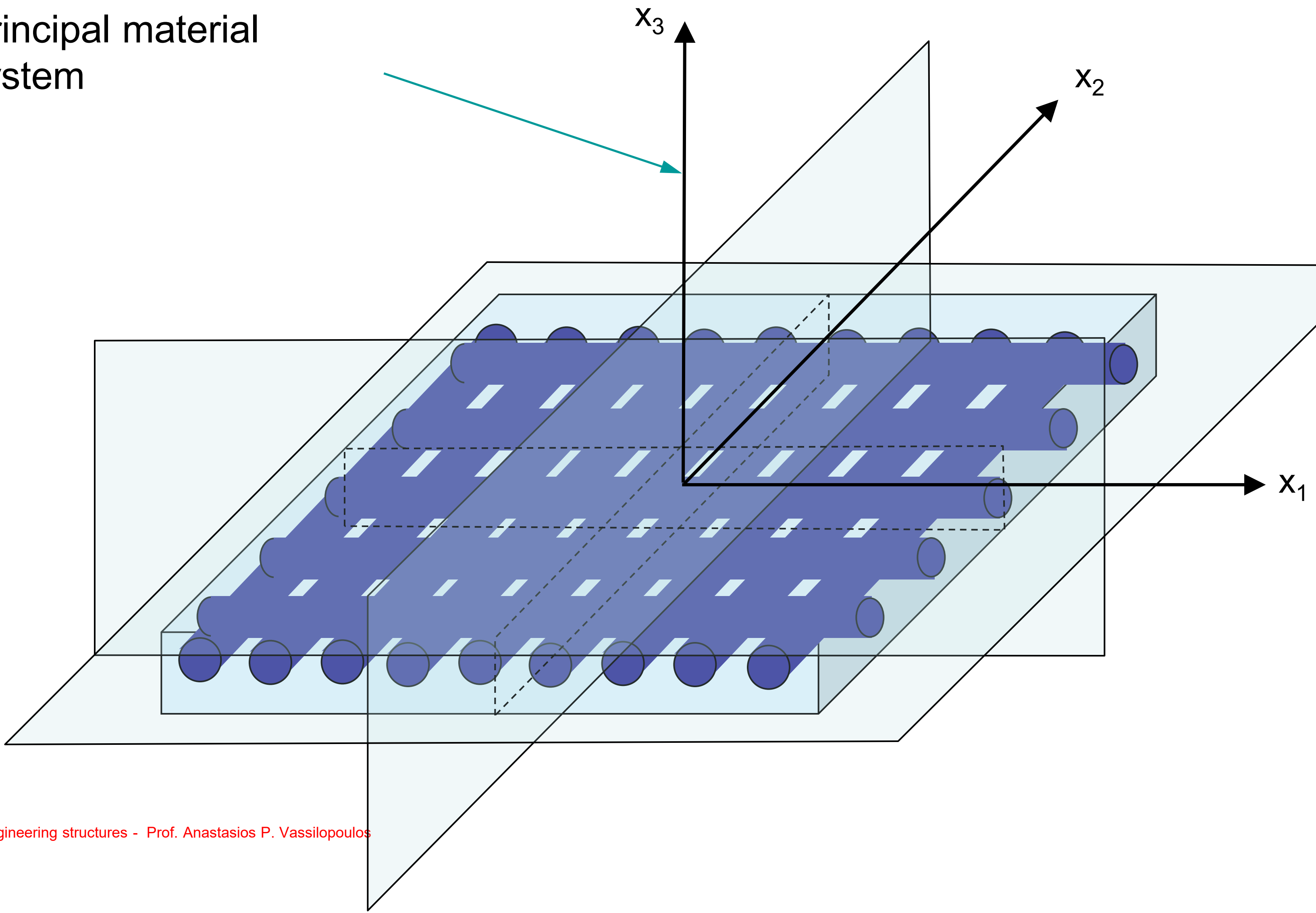


$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}.$$

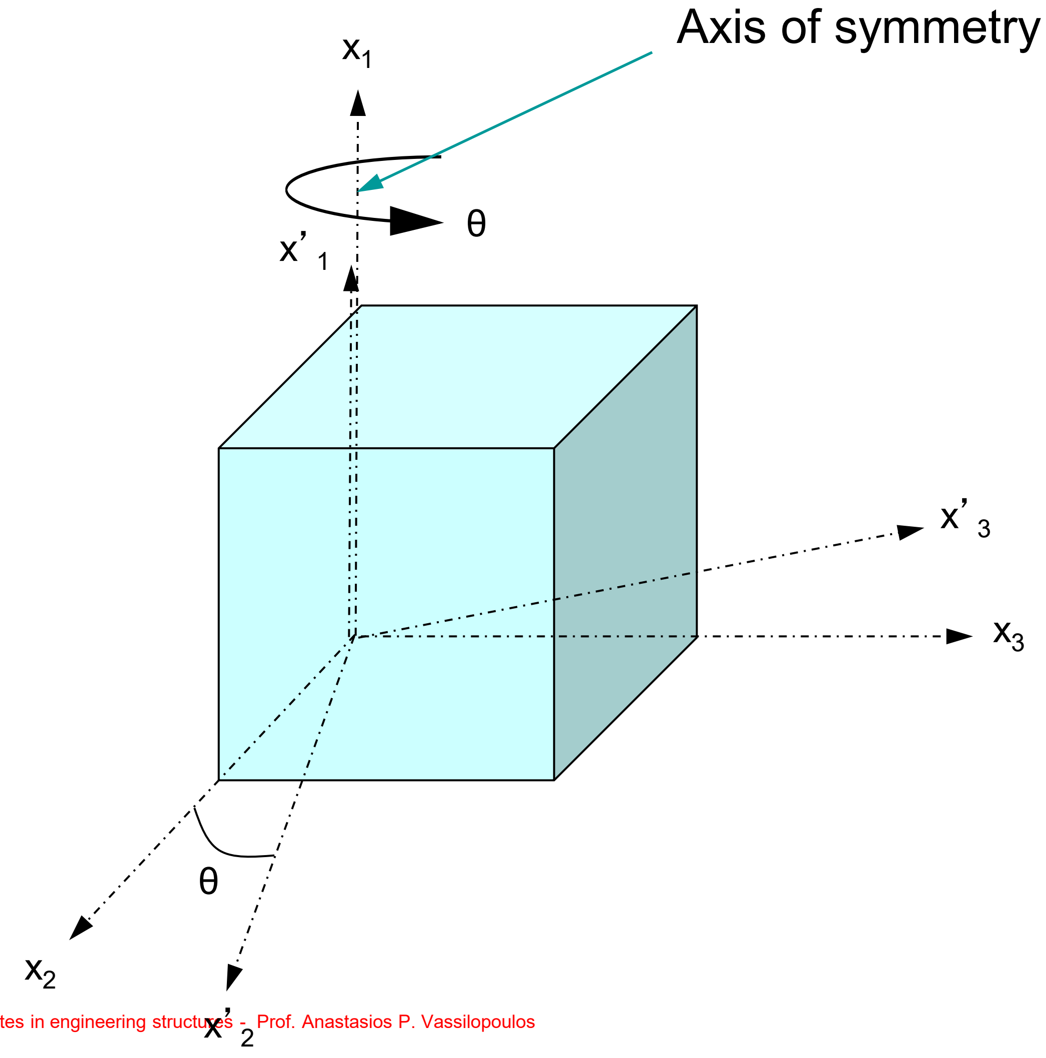
- In this case, only **9** independent variables are present in both stiffness and compliance matrices

Example of orthotropic material: Woven Fabric

Principal material system



Transversely isotropic material

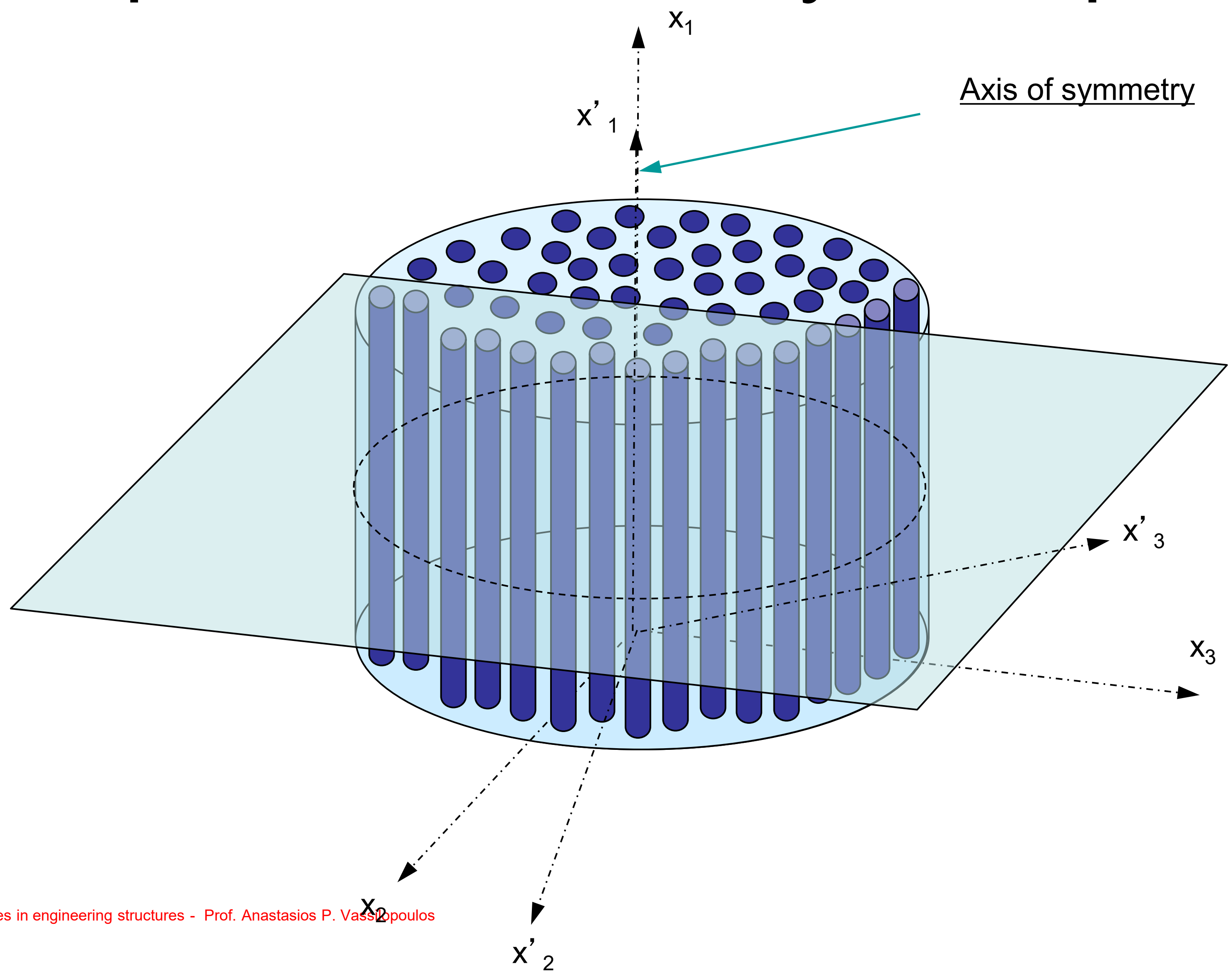


$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}.$$

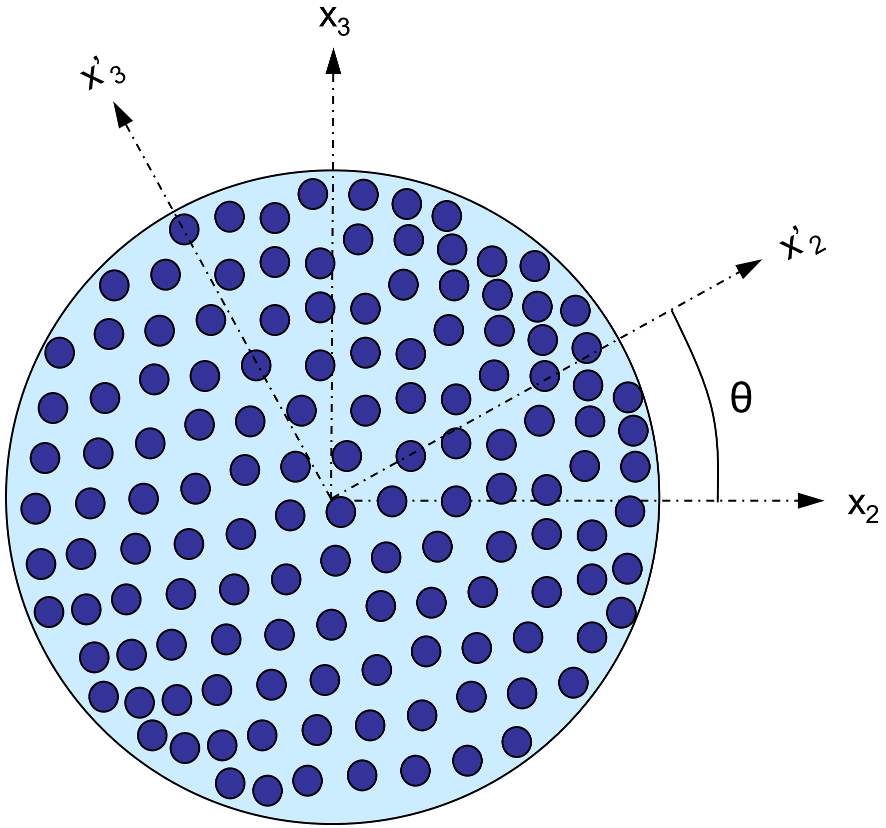
$$C_{22}=C_{33}, C_{12}=C_{13}, C_{55}=C_{66}, C_{44}=\frac{C_{22}-C_{23}}{2}.$$

In this case only **5** independent variables exist.

Example of transversely isotropic material

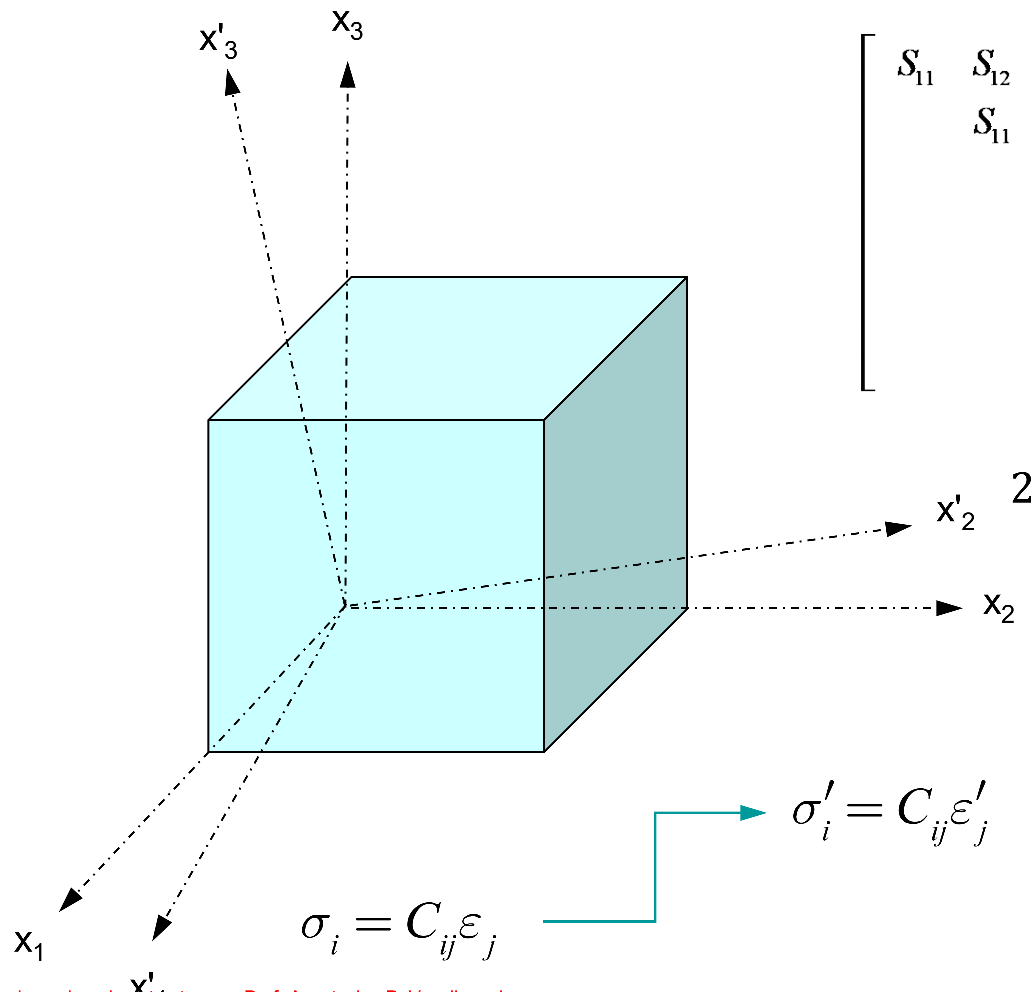


Isotropic plane of the transversely isotropic material



$$
\begin{bmatrix}
S'_{11} & S'_{12} & S'_{12} & 0 & 0 & 0 \\
S'_{12} & S'_{22} & S'_{23} & 0 & 0 & 0 \\
S'_{12} & S'_{23} & S'_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 2\left(S'_{22}-S'_{23}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & S'_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & S'_{66}
\end{bmatrix}
=
\begin{bmatrix}
S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 2\left(S_{22}-S_{23}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & S_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{bmatrix}
$$

Isotropic material



$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{11} & S_{12} & 0 & 0 & 0 \\ & & S_{11} & 0 & 0 & 0 \\ & & & 2(S_{11}-S_{12}) & 0 & 0 \\ & & & & 2(S_{11}-S_{12}) & 0 \\ & & & & & 2(S_{11}-S_{12}) \end{bmatrix}$$

$$2(S_{11}-S_{12})=2\left(\frac{1}{E}+\frac{\nu}{\mathrm{E}}\right)=\frac{2(1+\nu)}{\mathrm{E}}=\frac{1}{G}$$

$$\sigma_i = C_{ij}\varepsilon_j \rightarrow \sigma'_i = C_{ij}\varepsilon'_j$$

~~Principal axes~~

Summary-Independent elastic constants for various types of materials

Anisotropic:	21
Monoclinic:	13
Orthotropic:	9
Transversely isotropic:	5
Isotropic:	2

Summary:

Anisotropic, 21

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix}$$

Monoclinic, 13

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & 0 & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ & & & S_{44} & S_{45} & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix}$$

Orthotropic, 9

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix}$$

Transversely isotropic, 5

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix}$$

Isotropic, 2

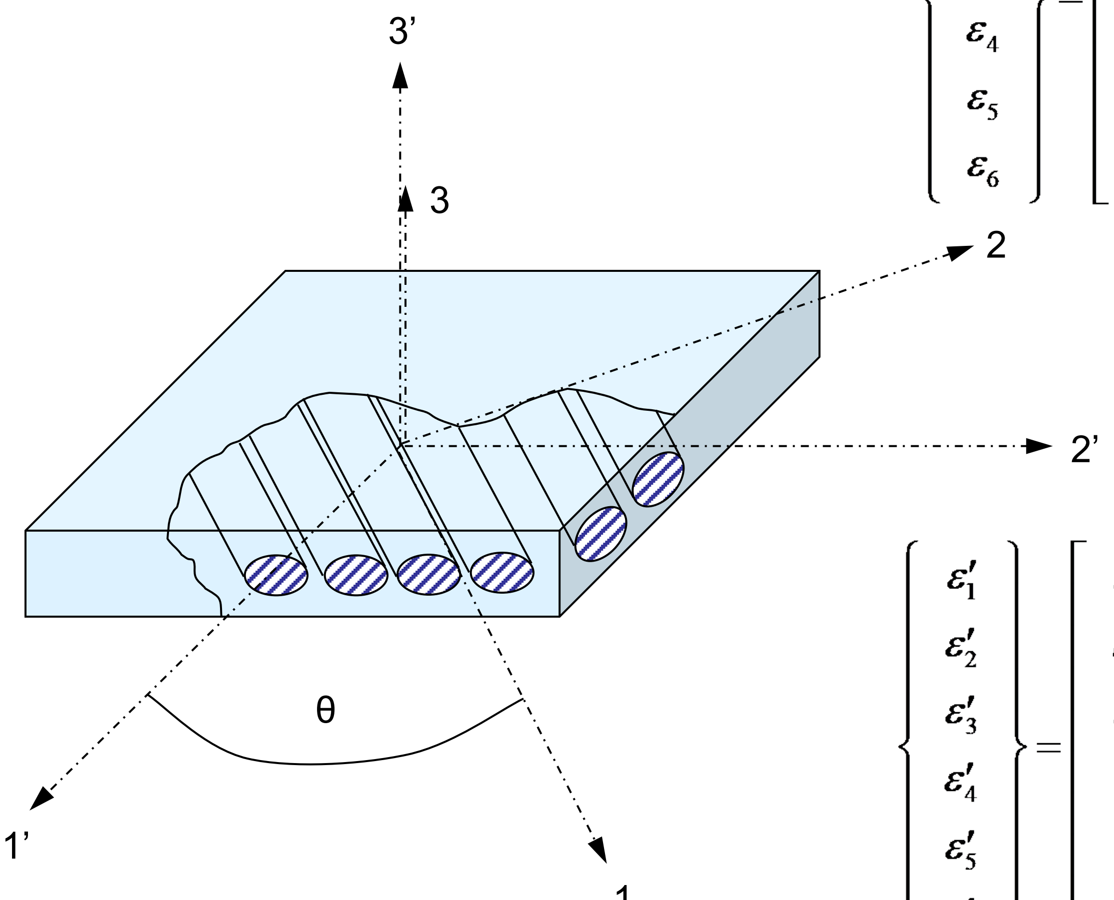
$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{11} & S_{12} & 0 & 0 & 0 \\ & & S_{11} & 0 & 0 & 0 \\ & & & 2(S_{11}-S_{12}) & 0 & 0 \\ & & & & 2(S_{11}-S_{12}) & 0 \\ & & & & & 2(S_{11}-S_{12}) \end{bmatrix}$$

The same is valid for C_{ij} components (…1/2 (C_{11}-C_{12})…)

Mechanical behavior of the orthotropic material

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix}=\begin{bmatrix}S_{11}&S_{12}&S_{13}&0&0&0\\ S_{12}&S_{22}&S_{23}&0&0&0\\ S_{13}&S_{23}&S_{33}&0&0&0\\ 0&0&0&S_{44}&0&0\\ 0&0&0&0&S_{55}&0\\ 0&0&0&0&0&S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}$$

on-axis orthotropic (9 parameters)

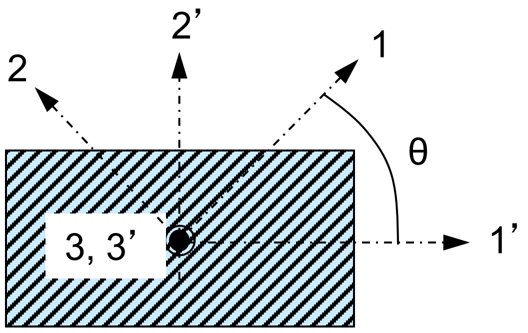


$$\begin{Bmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_3\\ \varepsilon'_4\\ \varepsilon'_5\\ \varepsilon'_6\end{Bmatrix}=\begin{bmatrix}S'_{11}&S'_{12}&S'_{13}&0&0&S'_{16}\\ S'_{12}&S'_{22}&S'_{23}&0&0&S'_{26}\\ S'_{13}&S'_{23}&S'_{33}&0&0&S'_{36}\\ 0&0&0&S'_{44}&S'_{45}&0\\ 0&0&0&S'_{45}&S'_{55}&0\\ S'_{16}&S'_{26}&S'_{36}&0&0&S'_{66}\end{bmatrix}\begin{Bmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_3\\ \sigma'_4\\ \sigma'_5\\ \sigma'_6\end{Bmatrix}$$

off-axis orthotropic (13 parameters)

Unlike with transversely isotropic materials…

Along the principal material axes, on-axis, of an orthotropic medium, there are **9** stiffness/compliance components that should be defined experimentally, while along the off-axis system there are **13**.
(However, they are not independent…)



$$S'_{11} = S_{11}m^4 + m^2n^2\left(2S_{12} + S_{66}\right) + S_{22}n^4$$

$$S'_{12} = m^2n^2\left(S_{11} + S_{22} - S_{66}\right) + S_{12}\left(m^4 + n^4\right)$$

$$S'_{13} = S_{13}m^2 + S_{23}n^2$$

$$S'_{16} = mn\left[2S_{11}m^2 - 2S_{22}n^2 - \left(2S_{12} + S_{66}\right)\left(m^2 - n^2\right)\right]$$

$$S'_{22} = S_{11}n^4 + m^2n^2\left(2S_{12} + S_{66}\right) + S_{22}m^4$$

$$S'_{23} = S_{13}n^2 + S_{23}m^2$$

$$S'_{26} = mn\left[2S_{11}n^2 - 2S_{22}m^2 + \left(2S_{12} + S_{66}\right)\left(m^2 - n^2\right)\right]$$

$$S'_{33} = S_{33}$$

$$S'_{36} = 2\left(S_{13} - S_{23}\right)mn$$

$$S'_{44} = S_{44}m^2 + S_{55}n^2$$

$$S'_{45} = \left(S_{55} - S_{44}\right)mn$$

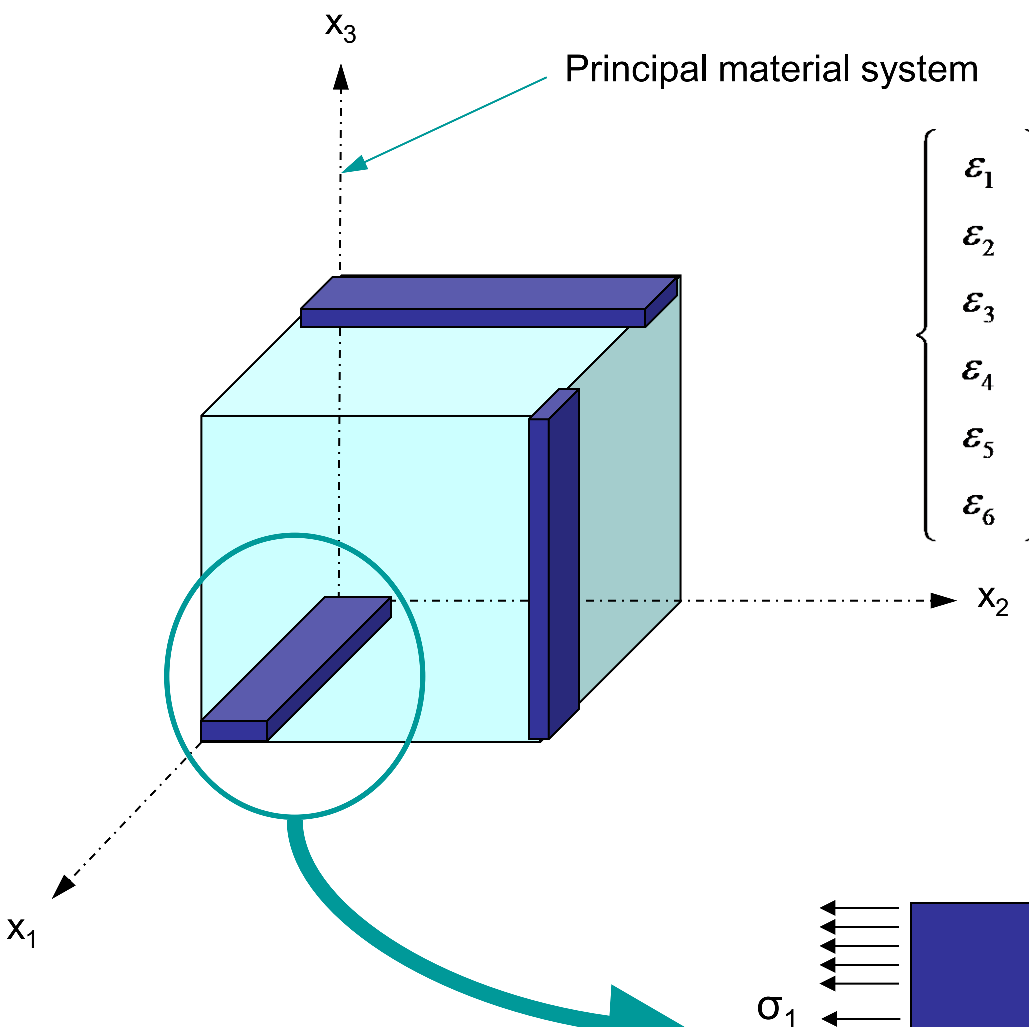
$$S'_{55} = S_{44}n^2 + S_{55}m^2$$

$$S'_{66} = 2\left(2S_{11} + 2S_{22} - 4S_{12}\right)m^2n^2 + S_{66}\left(m^2 - n^2\right)^2$$

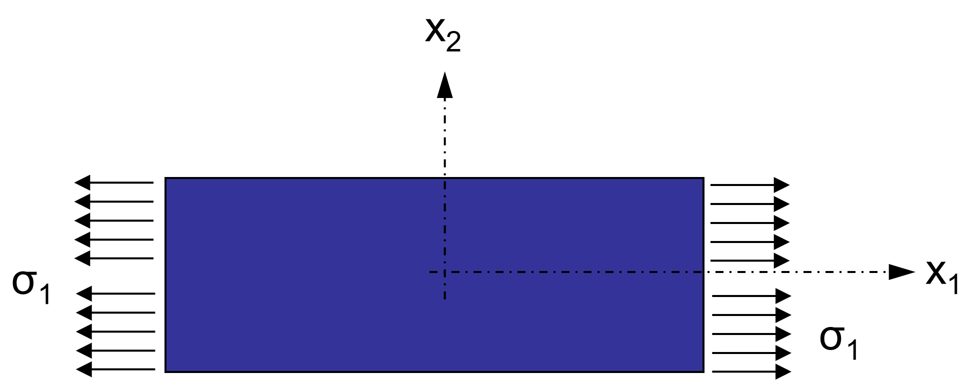
$$S'_{14} = S'_{15} = S'_{24} = S'_{25} = S'_{34} = S'_{35} = S'_{46} = S'_{56} = 0$$

Therefore, the experimental characterization of an orthotropic medium should be performed for the on-axis system, since the number of variables to be determined is limited to 9!!!

Technical elastic constants for the orthotropic medium



$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

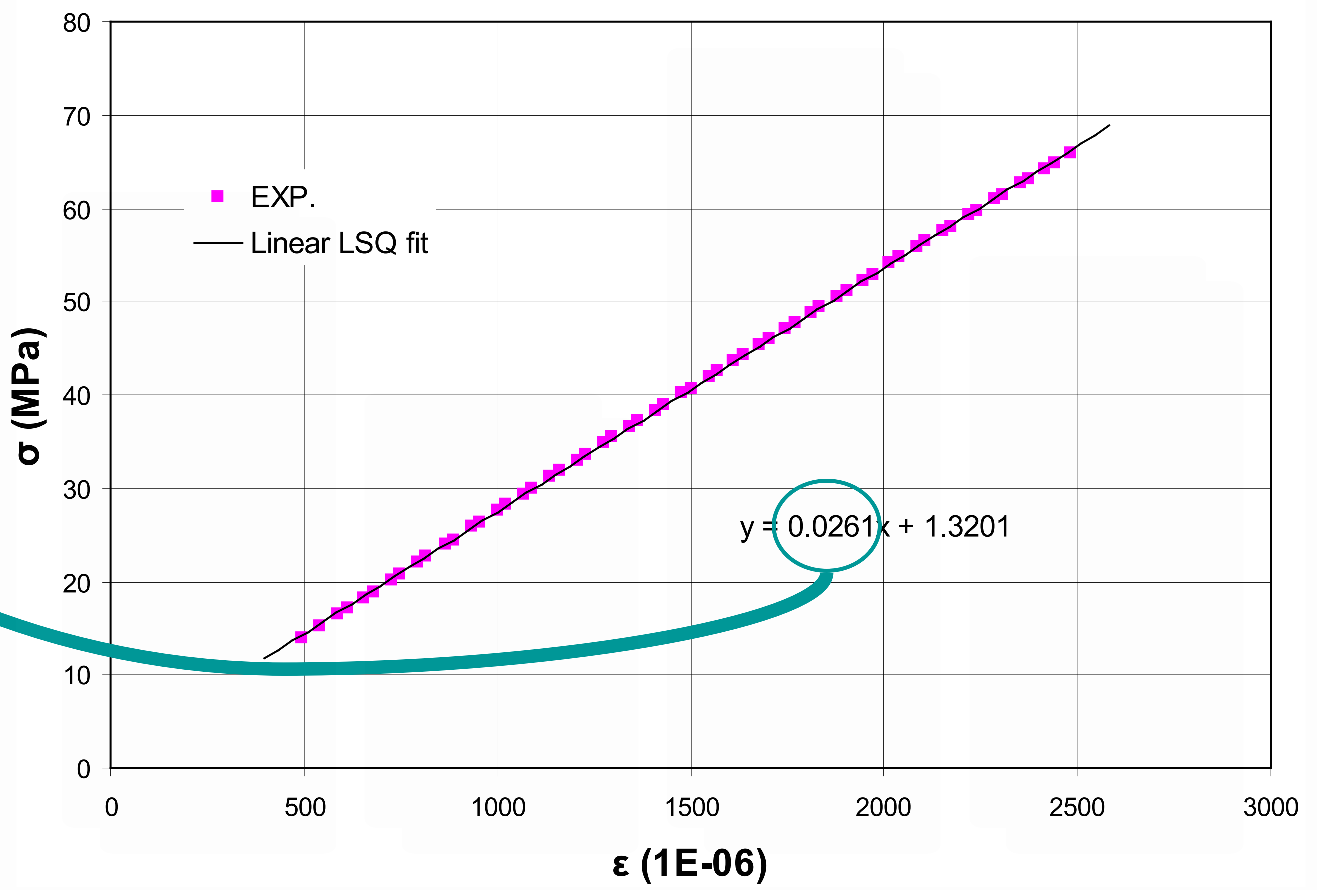


$$\varepsilon_1 = S_{11}\sigma_1$$

$$\varepsilon_1 = S_{11}\sigma_1$$

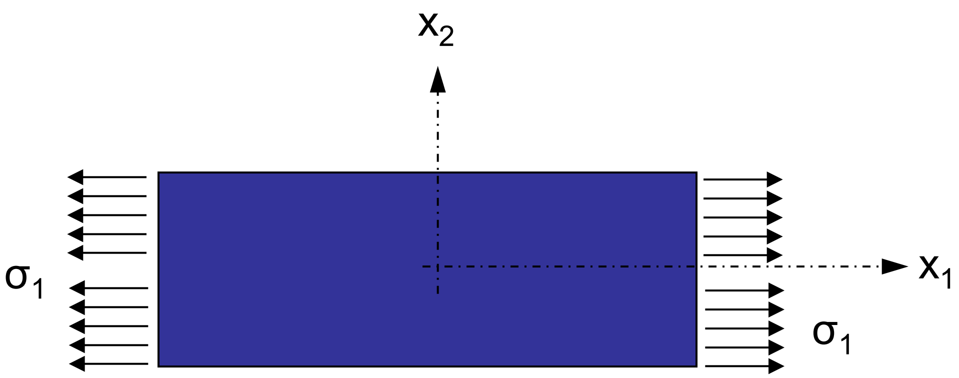
$$\frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}} = E_1$$

$$E_1 = 26.1 \text{ GPa}$$



By performing similar tests along the two other material directions we can define the other two elastic moduli

$$S_{22} = \frac{1}{E_2} \qquad S_{33} = \frac{1}{E_3}$$



$$\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{Bmatrix}=\begin{bmatrix}S_{11}&S_{12}&S_{13}&0&0&0\\S_{21}&S_{22}&S_{23}&0&0&0\\S_{31}&S_{32}&S_{33}&0&0&0\\0&0&0&S_{44}&0&0\\0&0&0&0&S_{55}&0\\0&0&0&0&0&S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\0\\0\\0\\0\\0\end{Bmatrix}$$

$$\varepsilon_1=S_{11}\sigma_1$$

$$\varepsilon_2=S_{21}\sigma_1$$

$$\varepsilon_3=S_{31}\sigma_1$$

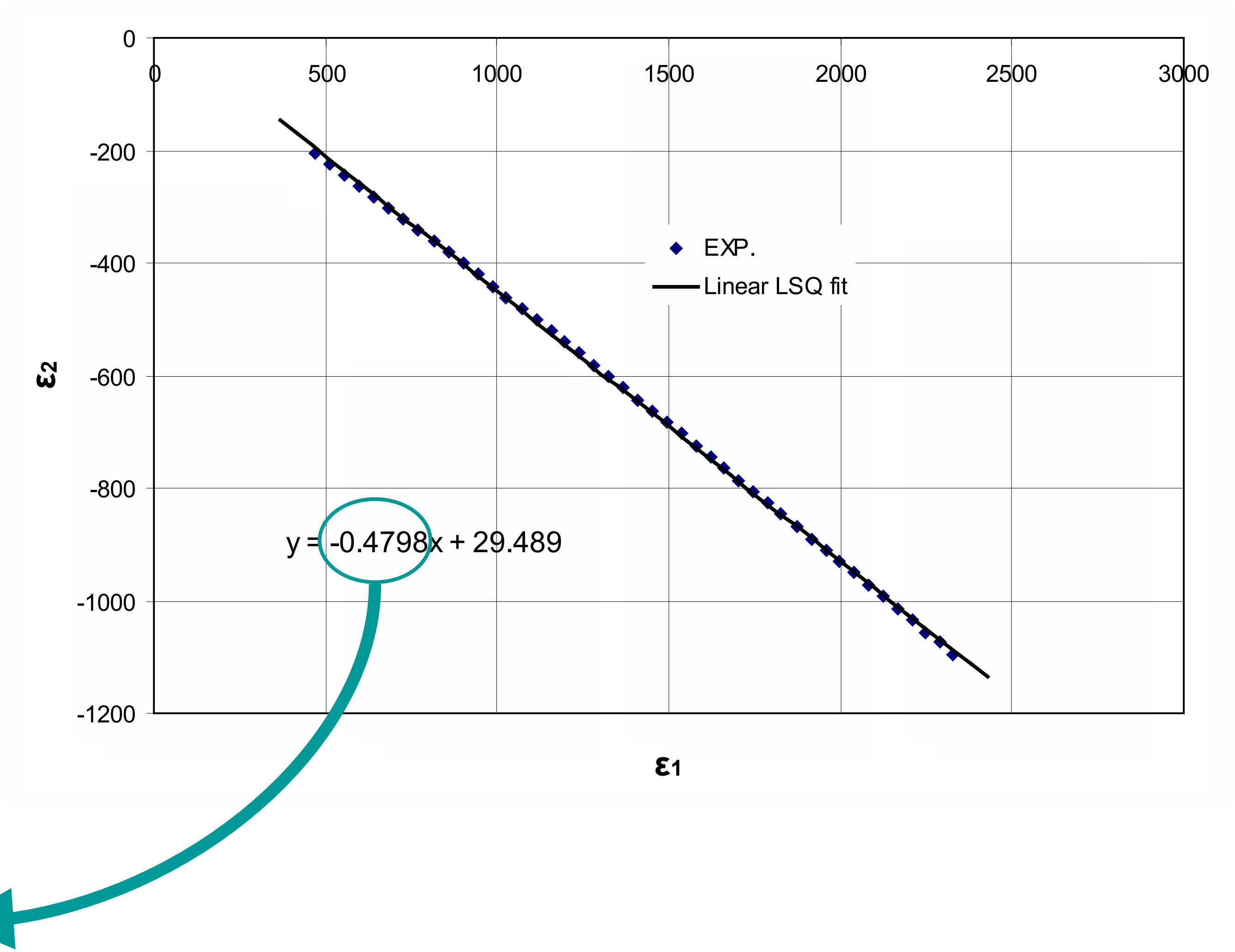
$$\frac{\varepsilon_2}{\varepsilon_1}=\frac{S_{21}}{S_{11}} \rightarrow \frac{\varepsilon_2}{\varepsilon_1}=\frac{S_{21}}{1/E_1} \rightarrow \frac{\varepsilon_2}{\varepsilon_1}=\frac{-\nu_{12}/E_1}{1/E_1} \rightarrow \nu_{12}=\frac{-\varepsilon_2}{\varepsilon_1}$$

$$\frac{\varepsilon_3}{\varepsilon_1}=\frac{S_{31}}{S_{11}} \rightarrow \frac{\varepsilon_3}{\varepsilon_1}=\frac{S_{31}}{1/E_1} \rightarrow \frac{\varepsilon_3}{\varepsilon_1}=\frac{-\nu_{13}/E_1}{1/E_1} \rightarrow \nu_{13}=\frac{-\varepsilon_3}{\varepsilon_1}$$

Therefore: $S_{21}=\frac{-\nu_{12}}{E_1}$ $\quad S_{31}=\frac{-\nu_{13}}{E_1}$

Experimental derivation of the Poisson ratio, ν_{12}

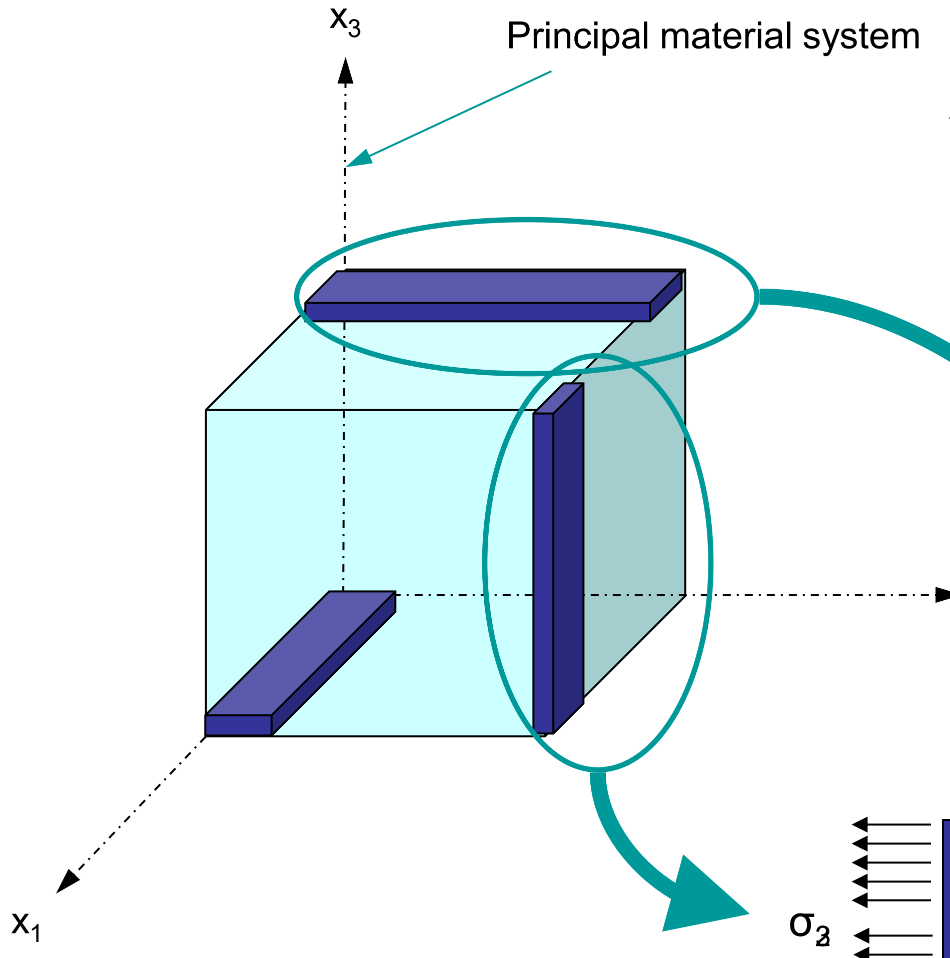
$$\nu_{12} = \frac{-\varepsilon_2}{\varepsilon_1}$$

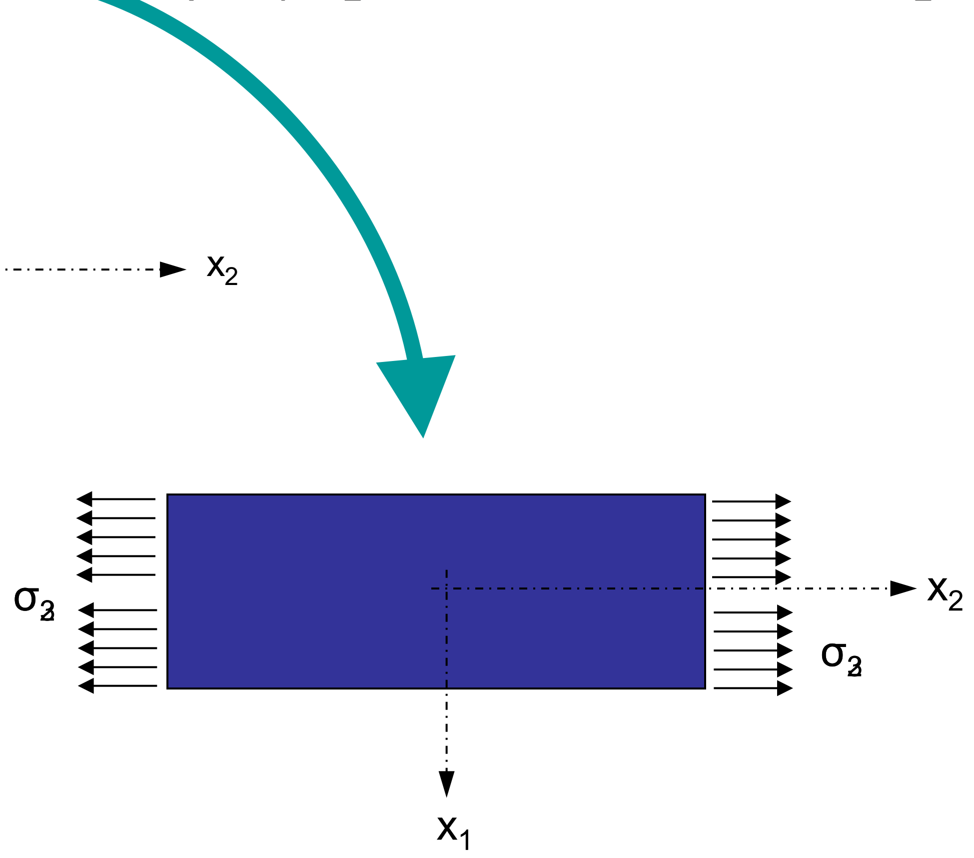


$$\nu_{12} = 0.4798$$

Similarly, other Poisson ratios are derived:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$





$$S_{13} = \frac{-\nu_{31}}{E_3} \qquad S_{23} = \frac{-\nu_{32}}{E_3}$$

Summing up:

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

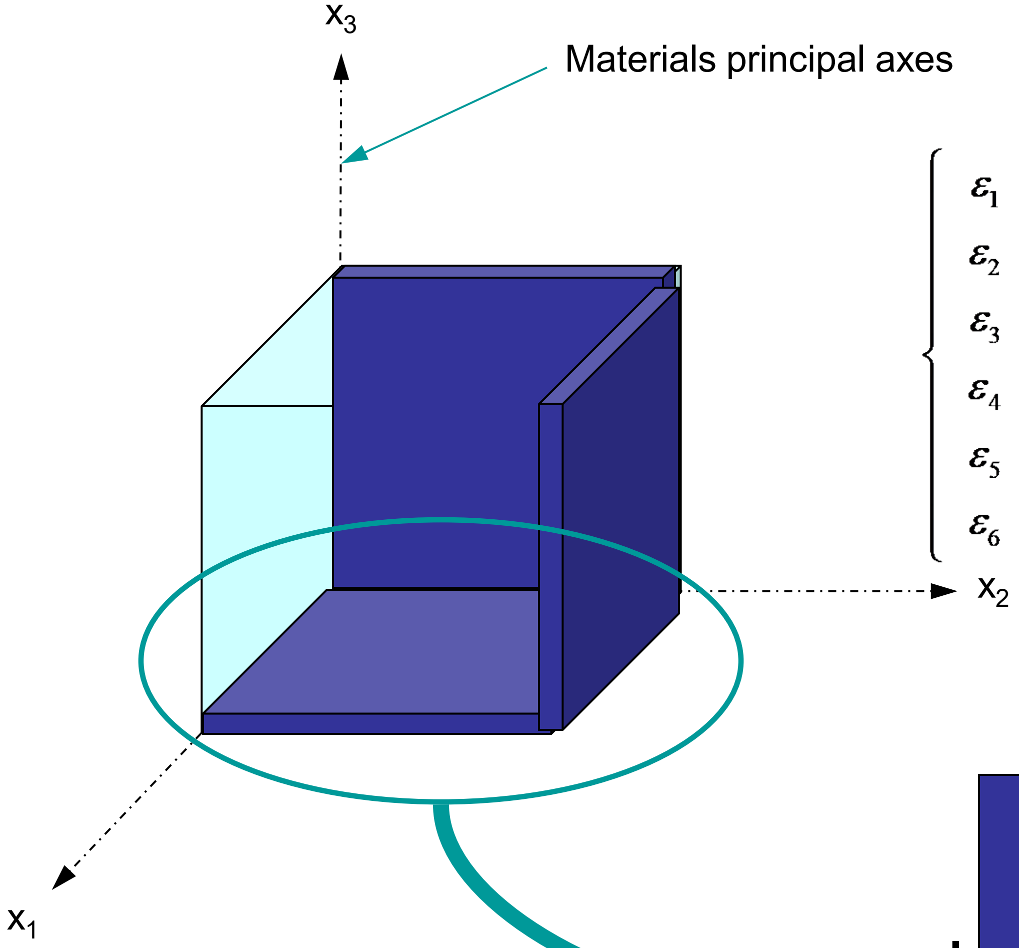
E_i : Young modulus along i-direction
v_{ij} : Poisson ratio: Normal train along j-direction due to uniaxial tensile loading along the i-direction.
e.g., $v_{ij} = -\varepsilon_j/\varepsilon_i$ when all components of stress vector are 0 except, σ_i.

Symmetry of **S** matrix poses that: $$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad , \qquad i,j=1,\ldots,3$$

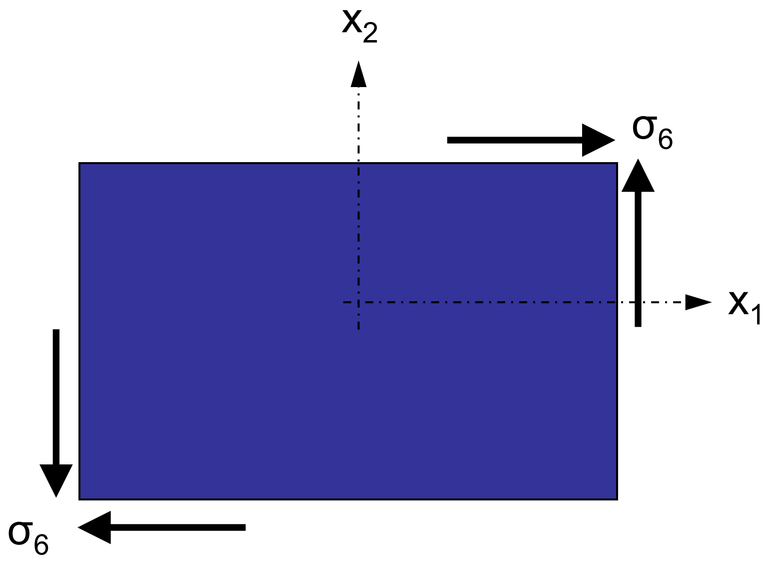
The rest of the compliance components are derived as follows:

$$S_{66} = \frac{1}{G_{12}}$$

$$\varepsilon_6 = S_{66}\sigma_6$$



$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_6 \end{Bmatrix}$$



Finally: The engineering constants of the orthotropic material are given by:

$$
\left[S_{ij}\right]=\begin{bmatrix}
\frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\
\frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\
\frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}}
\end{bmatrix}
$$

$$E_1=\frac{1}{S_{11}}$$

$$E_2=\frac{1}{S_{22}}$$

$$E_3=\frac{1}{S_{33}}$$

$$G_{12}\equiv\frac{\tau_{12}}{\gamma_{12}}=\frac{1}{S_{66}}$$

$$G_{23}\equiv\frac{\tau_{23}}{\gamma_{23}}=\frac{1}{S_{44}}$$

$$G_{13}\equiv\frac{\tau_{13}}{\gamma_{13}}=\frac{1}{S_{55}}$$

$$v_{12}\equiv-\frac{\varepsilon_2}{\varepsilon_1}=-\frac{S_{12}}{S_{11}}$$

$$v_{23}\equiv-\frac{\varepsilon_3}{\varepsilon_2}=-\frac{S_{23}}{S_{22}}$$

$$v_{13}\equiv-\frac{\varepsilon_3}{\varepsilon_1}=-\frac{S_{13}}{S_{11}}$$

- E_i: Young modulus, or elastic modulus
- G_{ij}: Shear modulus
- v_{ij}: Poisson ratio

Since S (the compliance matrix) is the transverse of C, (the stiffness matrix):

$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3}$$

$$C_{11} = \frac{1 - \nu_{23}\nu_{32}}{E_2 E_3 \Delta}$$

$$C_{12} = \frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2 E_3 \Delta} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1 E_3 \Delta}$$

$$C_{13} = \frac{\nu_{31} + \nu_{21}\nu_{32}}{E_2 E_3 \Delta} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{E_1 E_2 \Delta}$$

$$C_{22} = \frac{1 - \nu_{13}\nu_{31}}{E_1 E_3 \Delta}$$

$$C_{23} = \frac{\nu_{32} + \nu_{12}\nu_{31}}{E_1 E_3 \Delta} = \frac{\nu_{23} + \nu_{21}\nu_{13}}{E_1 E_2 \Delta}$$

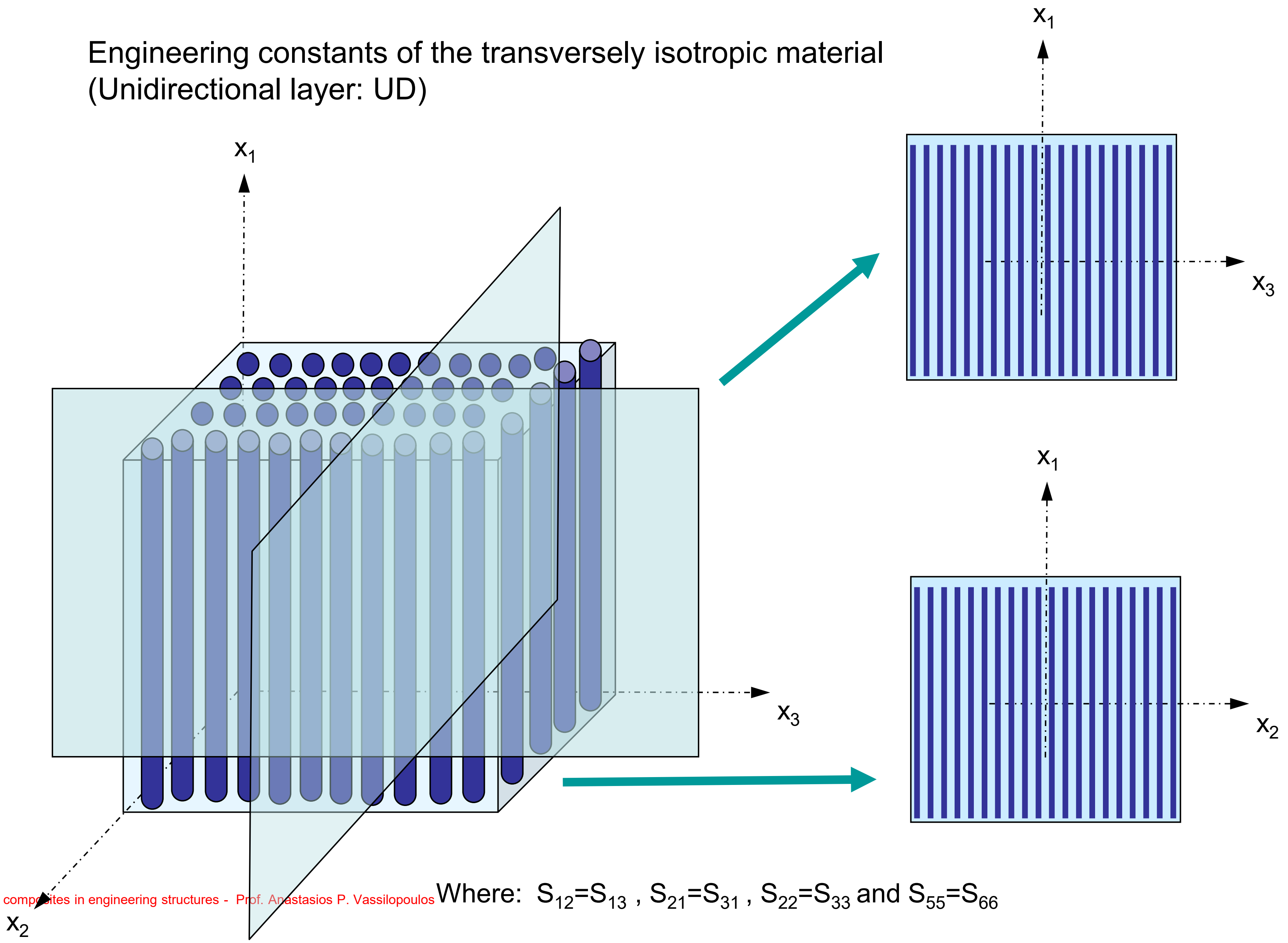
$$C_{33} = \frac{1 - \nu_{12}\nu_{21}}{E_1 E_2 \Delta}$$

$$C_{44} = G_{23}$$

$$C_{55} = G_{13}$$

$$C_{66} = G_{12}$$

Engineering constants of the transversely isotropic material (Unidirectional layer: UD)



Where: $S_{12}=S_{13}$, $S_{21}=S_{31}$, $S_{22}=S_{33}$ and $S_{55}=S_{66}$

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{21}}{E_2} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_2} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{23})}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

Only **5** constants should be derived:

- E_1: Tensile modulus in the fiber direction
- E_2: Tensile modulus in the transverse dir.
- G_{12}: Shear modulus
- v_{12}: Major Poisson ratio (v_{21}: Minor Poisson ratio)
- v_{23}: Transverse Poisson ratio

  or

- G_{23}: Transverse shear modulus

Typical values of engineering constants for orthotropic materials

Commercial name	Material	E_1 [GPa]	E_2 [GPa]	G_{12} [GPa]	ν_{12}	v_f
T300/5208	Gr/Ep	181	10.3	7.17	0.28	0.70
B(4)/5505	B/Ep	204	18.5	5.59	0.23	0.50
AS/3501	Gr/Ep	138	8.96	7.1	0.30	0.66
Scotchply 1002 (3M)	Gl/Ep	38.6	8.27	4.14	0.26	0.45
Kevlar 49	Ar/Ep	76	5.5	2.3	0.34	0.60
OPTIMAT UD	Gl/Ep	39.01	15.15	5.83	0.29	0.52
SISTEMA ±45	Gr/Ep	5.542	5.542	12.887	0.810	0.55
GEO UD	GRP	25.12	8.576	2.284	0.215	0.40

Case study

- Define the compliance matrix for a graphite/epoxy lamina with the following material properties
  - E_1=181 GPa, E_2=10.3 GPa, E_3=10.3 GPa
  - v_{12}=0.28, v_{23}=0.60, v_{13}=0.27
  - G_{12}=7.17 GPa, G_{23}=3.0 GPa, G_{31}=7.0 GPa

$$v_{21} = E_2 \frac{v_{12}}{E_1} \qquad v_{32} = E_3 \frac{v_{23}}{E_2} \qquad v_{31} = E_3 \frac{v_{13}}{E_1}$$

EPFL

Solution

$$S_{11} = \frac{1}{E_1} = \frac{1}{181\times10^9} = 5.525\times10^{-12}\,Pa^{-1}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3\times10^9} = 9.709\times10^{-11}\,Pa^{-1}$$

$$S_{33} = \frac{1}{E_3} = \frac{1}{10.3\times10^9} = 9.709\times10^{-11}\,Pa^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181\times10^9} = -1.547\times10^{-12}\,Pa^{-1}$$

$$S_{13} = -\frac{\nu_{13}}{E_1} = -\frac{0.27}{181\times10^9} = -1.492\times10^{-12}\,Pa^{-1}$$

$$S_{23} = -\frac{\nu_{23}}{E_2} = -\frac{0.6}{10.3\times10^9} = -5.825\times10^{-11}\,Pa^{-1}$$

$$S_{44} = \frac{1}{G_{23}} = \frac{1}{3\times10^9} = 3.333\times10^{-10}\,Pa^{-1}$$

$$S_{55} = \frac{1}{G_{31}} = \frac{1}{7\times10^9} = 1.429\times10^{-10}\,Pa^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17\times10^9} = 1.395\times10^{-10}\,Pa^{-1}.$$

$$\begin{bmatrix} 5.525\times10^{-12} & -1.547\times10^{-12} & -1.492\times10^{-12} & 0 & 0 & 0 \\ -1.547\times10^{-12} & 9.709\times10^{-11} & -5.825\times10^{-11} & 0 & 0 & 0 \\ -1.492\times10^{-12} & -5.825\times10^{-11} & 9.709\times10^{-11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.333\times10^{-10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.429\times10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.395\times10^{-10} \end{bmatrix} Pa^{-1}$$

$$[C]=[S]^{-1}$$

$$[C]=$$

$$\begin{bmatrix} 0.1850\times10^{12} & 0.7269\times10^{10} & 0.7204\times10^{10} & 0 & 0 & 0 \\ 0.7269\times10^{10} & 0.1638\times10^{11} & 0.9938\times10^{10} & 0 & 0 & 0 \\ 0.7204\times10^{10} & 0.9938\times10^{10} & 0.1637\times10^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3000\times10^{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6998\times10^{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7168\times10^{10} \end{bmatrix} Pa$$

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta}$$

$$\Delta = \left(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{13}\nu_{31}-2\nu_{21}\nu_{32}\nu_{13}\right)/\left(E_1E_2E_3\right)$$

Solution

$$S_{11} = \frac{1}{E_1} = \frac{1}{181\times10^9} = 5.525\times10^{-12}\,Pa^{-1}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3\times10^9} = 9.709\times10^{-11}\,Pa^{-1}$$

$$S_{33} = \frac{1}{E_3} = \frac{1}{10.3\times10^9} = 9.709\times10^{-11}\,Pa^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181\times10^9} = -1.547\times10^{-12}\,Pa^{-1}$$

$$S_{13} = -\frac{\nu_{13}}{E_1} = -\frac{0.27}{181\times10^9} = -1.492\times10^{-12}\,Pa^{-1}$$

$$S_{23} = -\frac{\nu_{23}}{E_2} = -\frac{0.6}{10.3\times10^9} = -5.825\times10^{-11}\,Pa^{-1}$$

$$S_{44} = \frac{1}{G_{23}} = \frac{1}{3\times10^9} = 3.333\times10^{-10}\,Pa^{-1}$$

$$S_{55} = \frac{1}{G_{31}} = \frac{1}{7\times10^9} = 1.429\times10^{-10}\,Pa^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17\times10^9} = 1.395\times10^{-10}\,Pa^{-1}\,.$$

$$\begin{bmatrix} 5.525\times10^{-12} & -1.547\times10^{-12} & -1.492\times10^{-12} & 0 & 0 & 0 \\ -1.547\times10^{-12} & 9.709\times10^{-11} & -5.825\times10^{-11} & 0 & 0 & 0 \\ -1.492\times10^{-12} & -5.825\times10^{-11} & 9.709\times10^{-11} & 0 & 0 & \text{[illegible]} \\ 0 & 0 & 0 & 3.333\times10^{-10} & \text{[illegible]} & \text{[illegible]} \\ 0 & 0 & 0 & 0 & \text{[illegible]} & \text{[illegible]} \\ 0 & 0 & 0 & 0 & \text{[illegible]} & \text{[illegible]}\times10^{-10} \end{bmatrix} Pa^{-1}$$

Calculate the stiffness matrix for the same material @ 15°

$$\begin{bmatrix} 0.1850\times10^{12} & \text{[illegible]} & \text{[illegible]}\times10^{10} & 0 & 0 & 0 \\ 0.7269\text{[illegible]} & \text{[illegible]} & \text{[illegible]}38\times10^{10} & 0 & 0 & 0 \\ 0.7204\text{[illegible]} & \text{[illegible]} & 0.1637\times10^{11} & 0 & 0 & 0 \\ 0 & \text{[illegible]} & 0 & 0.3000\times10^{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6998\times10^{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7168\times10^{10} \end{bmatrix} Pa$$

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta}$$

$$\Delta = \left(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{13}\nu_{31}-2\nu_{21}\nu_{32}\nu_{13}\right)/\left(E_1E_2E_3\right)$$